Math 623 Exam 1 Solutions

1. Suppose that $A \in M_n(\mathbb{F})$ has RREF of $I_n$. Prove that $A$ may be written as the product of elementary matrices.

See Ex. 0.18. We put $A$ into RREF using elementary matrices $E_1, \ldots, E_k$; i.e. $I = E_k E_{k-1} \cdots E_2 E_1 A$. We then multiply both sides by the same matrix (repeatedly) to get $E_k^{-1} E_{k-1}^{-1} \cdots E_2^{-1} E_1^{-1} A = I$. The last observation we need is that the inverse of an elementary matrix is elementary, so in fact $E_k^{-1}, \ldots, E_1^{-1}$ are each elementary matrices.

2. Let $J \in M_n(\mathbb{R})$ be the matrix all of whose entries are 1. Find $\sigma(J)$, and for each eigenvalue find a basis for the corresponding eigenspace.

See Ex 1.1.5. Set $e = (1,1,\ldots,1)$; we have $Je = ne$, so $(n,e)$ is an eigenvalue-eigenvector pair. Set $x_i = e - ne_i$; we have $Jx_i = 0 = 0x_i$, so $(0,x_i)$ is an eigenvalue-eigenvector pair. However $\{x_1, \ldots, x_n\}$ is too big (it is dependent, since the sum is zero). Any subset of size $n - 1$ will be a basis for the eigenspace corresponding to eigenvalue 0. Note: there are no other eigenvalues since the ones we have found already have total multiplicity $n$.

3. Give an example of a matrix $M \in M_3(\mathbb{C})$ that is diagonalizable but not diagonal, and has fewer than 3 distinct eigenvalues.

See Ex. 1.3.9. The simplest approach is to start with a diagonal matrix $\Lambda$, then calculate $S \Lambda S^{-1}$ – this is guaranteed to be diagonalizable. We can try something like $\Lambda = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, which yields $M = S \Lambda S^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

4. Calculate the adjugate and eigenvalues of $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{bmatrix}$.

See Ex. 0.31. This is a straightforward computation; $\text{adj} A = \begin{bmatrix} 1 & -2 & -4 \\ 1 & 1 & -4 \\ 1 & 1 & 1 \end{bmatrix}$, $\sigma(B) = \{-1, 0, 5\}$.

5. Set $P_3(t)$ to be the set of all polynomials of degree at most 3, in variable $t$, with real coefficients. Find the rank and nullity of linear transformation $T : P_3(t) \rightarrow P_3(t)$ given by $T(f(t)) = t \frac{df(t)}{dt}$.

See Ex. 0.10. A basis for $P_3(t)$ is $\{1, t, t^2, t^3\}$, and we calculate $T(1) = 0, T(t) = t, T(t^2) = 2t^2, T(t^3) = 3t^3$. Hence the range of $T$ is spanned by $\{t, 2t^2, 3t^3\}$; these are clearly linearly independent, hence the rank of $T$ is 3. By the rank-nullity theorem, the rank plus the nullity is the dimension of $P_3(t)$, namely 4. Hence the nullity of $T$ is 1.

6. A matrix $A \in M_3(\mathbb{C})$ is a square root of $B$ if $A^2 = B$. Prove that every diagonalizable $B \in M_3(\mathbb{C})$ has a square root.

See 1.3.P7. Suppose $B$ is diagonalizable; then there is invertible $S$ where $B = SDS^{-1}$, where $D = \text{diag}(a,b,c)$. Now, set $E = \text{diag}(\sqrt{a}, \sqrt{b}, \sqrt{c})$ (choose either square root if ambiguous), and $A = SES^{-1}$. We calculate $A^2 = SES^{-1}SES^{-1} = SE^2 S^{-1} = SDS^{-1} = B$.

[Note: $\mathbb{C}$ is necessary, else we might not be able to take square roots.]

7. Let $A \in M_3(\mathbb{C})$ be skew-symmetric. Prove that $P_A(t) = -P_A(-t)$, and that if $\lambda$ is an eigenvalue of $A$, so is $-\lambda$.

See 1.4.P2. Since $A$ is skew-symmetric, we have $A = -A^T$. We calculate $P_A(t) = \det(tI - A) = \det(tI - (-A^T)) = (-1)^3 \det(-tI - A^T) = -\det((-tI - A^T)^T) = -\det(-tI - A) = -P_A(-t)$. Suppose now that $(\lambda, x)$ is a (right) eigenvalue-eigenvector pair for $A$. Then $Ax = \lambda x$. We take transposes to get $x^T A^T = \lambda x^T$, then negate to get $x^T (-A^T) = (-\lambda)x^T$ or $x^T A = (-\lambda)x^T$. Hence $(-\lambda, x)$ is a (left) eigenvalue-eigenvector pair for $A$, and hence $-\lambda$ is an eigenvalue for $A$. 
