1. Suppose that $S$ is a vector space over field $F$, and $S_1, S_2$ are both subspaces of $S$. Prove that $S_1 + S_2$ is a subspace of $S$.

2. Suppose that $S$ is a vector space over field $F$, and $S_1, S_2$ are both subspaces of $S$. Prove that $S_1 ∩ S_2$ is a subspace of $S$.

3. Consider the vector space $\mathbb{R}^2$. Find two subspaces $S_1, S_2$ such that $S_1 ∪ S_2$ is not a subspace.

4. Suppose that matrix $A ∈ M_{m,n}(F)$, prove that the rowspace and nullspace are both subspaces of $F^n$.

5. Prove that every superlist of a dependent list of vectors is again dependent.

6. For matrix $A ∈ M_{m,n}(F)$, prove that the rowspace and nullspace are both subspaces of $F^n$.

7. Find an infinite-dimensional vector space $V$, with two proper nontrivial subspaces $V_1, V_2$ such that $V_1$ is finite-dimensional and $V_2$ is infinite-dimensional.

8. Set $P_2(t)$ to be the set of all polynomials of degree at most 2, in variable $t$, with real coefficients. Prove that $P_2(t)$ is isomorphic to $\mathbb{R}^3$.

9. Let $P_2(t)$ be as in (8). Prove that $T : P_2(t) → P_2(t)$ given by $T(f(t)) = t \frac{df(t)}{dt}$ is a linear transformation.

10. Let $T$ be as in (9). Find its rank and nullity.

11. For matrices $A, B$ where $AB$ is defined, prove that $(AB)^T = B^T A^T$ and $(AB)^* = B^* A^*$.

12. For complex-valued matrix $A = [a_{ij}]$, prove that $A^* = A^T$ if and only if $a_{ij} ∈ \mathbb{R}$ for all $i, j$.

13. For complex-valued matrix $A = [a_{ij}]$, prove that $A + A^T$ is symmetric, $A + \overline{A}$ is real, and $A + A^*$ is Hermitian.

14. Calculate the determinant and permanent of $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

15. Calculate the inverse of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

16. Prove that the inverse of an elementary matrix is elementary.

17. For $A, B ∈ M_n(F)$, prove that $AB$ is invertible if and only if both $A, B$ are invertible.

18. Suppose that square matrix $A$ has RREF of $I$. Prove that $A$ may be written as the product of elementary matrices.

19. If $A ∈ M_{m,n}(F)$, prove that $\text{rank} A ≤ \min(m, n)$.

20. If $A ∈ M_{m,n}(F)$, and $B ∈ M_{n,n}(F)$, prove that $\text{rank} A ≥ \text{rank} AB$.

21. If $A ∈ M_{m,n}(F)$, and $B ∈ M_{n,n}(F)$ is nonsingular, prove that $\text{rank} A = \text{rank} AB$.

22. If $A ∈ M_{m,n}(\mathbb{C})$, prove that $\text{rank} A = \text{rank} A^* A$.

23. Prove that if square matrix $A$ has a left inverse, then it also has a right inverse, and they are the same.

24. Let $V = \mathbb{C}^2$. Define $\langle x, y \rangle = y^* \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} x$. Prove that this defines an inner product on $V$.

25. With $V, \langle ·, · \rangle$ as in (24), calculate the angle between $x = (-1, 2)$ and $y = (1, 1)$.

26. With $V, \langle ·, · \rangle$ as in (24), use Gram-Schmidt starting with $\{e_1, e_2\}$ to find an orthonormal basis for $V$.

27. Suppose $S$ is a subspace of $\mathbb{C}^n$. Prove that $(S^⊥)^⊥ = S$.

28. Suppose $S_1, S_2$ are subspaces of $\mathbb{C}^n$. Prove that $(S_1 + S_2)^⊥ = S_1^⊥ ∩ S_2^⊥$.

29. Calculate $C_2(A)$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

30. Calculate $C_2(A^2)$ for $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, and verify that $C_2(A^2) = C_2(A)^2$.

31. Calculate the adjugate of $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. 