MATH601 Spring 2008 Handout 8: Dodgeball

Unit 4: Cardinals

At left is the board to play Dodgeball. It is a two-player game, between the Pitcher and the Dodger. They take turns writing in the boxes – at the end of the game each box will have either an X or an O. The Pitcher goes first, and fills the entire top row (with X's or O's, as desired). The Dodger goes second, and fills the first box of the bottom row. Then the Pitcher goes, and fills the entire second row. Then the Dodger goes, and fills the second box of the bottom row. After both players have taken six turns, we find out who wins. If the Dodger's row matches any of the top six rows, the Pitcher wins. If, however, the Dodger managed to dodge all six of the Pitcher's rows, the Dodger wins.

Exercises:

1. Which player can guarantee a win? Describe the winning strategy.
2. We change the game so that Dodger goes first. Which player can guarantee a win? Describe the winning strategy.
3. We change the game so that Pitcher gets an extra, seventh, row, that is played before the game starts (Pitcher starts by filling two rows). Which player can guarantee a win? Describe the winning strategy.
4. Suppose we had a bijection f between $\mathbb{N}$ and the real interval $(0, 1)$. Then $f(1) = 0.a_1a_2a_3a_4\cdots$, $f(2) = 0.b_1b_2b_3b_4\cdots$, and so on. Think of these as infinitely many turns that Pitcher has taken. Show that Dodger can find a winning response, a real number that is not in the range of f. This contradiction means that there is NO bijection, which means that these two sets are NOT equicardinal.

   Let S be any set, and let 2^S represent the power set of S, the set of all subsets of S. For example, if $S = \{A, B\}$, then $2^S = \{\{\}, \{A\}, \{B\}, \{A, B\}\}$.
5. Prove that if S is finite, then $|2^S| = 2^{|S|}$.
   Hint: Every element of 2^S is a subset of S; it either does or does not contain x, for every $x \in S$.
6. Prove that for any S, $|2^S| \neq |S|$. The interesting case is when S is not finite, because of the previous exercise.
   Proof sketch: Suppose there were a bijection $f : S \to 2^S$. Then each element $x \in S$ gets paired with $f(x)$, a subset of S. Color each element $x \in S$ either red or blue, by the following rule. If $x \in f(x)$, then color x red; if $x \notin f(x)$, then color x blue. Now, let y be the set of all blue elements of S. y is a subset of S, so it is an element of 2^S, so it should be in the range of f. There must be some $x \in S$ with $f(x) = y$. What color is x?

   This is a famous theorem of Cantor, and proves that there are not only different sizes of infinity, but *infinitely* many sizes of infinity, each larger than the last: $S, 2^S, 2^{2^S}, \ldots$.