MATH601 Spring 2008 Handout 5
Unit 2: Real Numbers

In this unit, we construct real numbers out of fractions. We take as our starting point the set of fractions $\mathbb{Q} = \{a/b : a \in \mathbb{Z}, b \in \mathbb{N}\}$, or more precisely the set of equivalence classes, where $[a/b] = [a'/b']$ iff $ab' = a'b$. You may assume all your 'common knowledge' about fractions; also, $\mathbb{Q}$ is a dense, Archimedean, ordered, field. All of these terms are defined below for your convenience.

Field: A field is a set of numbers that satisfies the following twelve axioms:

1. There is a closed operation $+$ (i.e. for every a, b in the field, $a + b$ is again in the field)
2. $+$ is commutative (i.e. for every a, b in the field, $a + b = b + a$)
3. $+$ is associative (i.e. for every a, b, c in the field, $a + (b + c) = (a + b) + c$)
4. $+$ has an identity, called 0 (i.e. for every a in the field, $a + 0 = a$)
5. $+$ has inverses (i.e. for every a in the field, there is some b in the field, with $a + b = 0$. We write $-a$ for this special b.)
6. There is a closed operation $\times$ (i.e. for every a, b in the field, $a \times b$ is again in the field)
7. $\times$ is commutative (i.e. for every a, b in the field, $a \times b = b \times a$)
8. $\times$ is associative (i.e. for every a, b, c in the field, $a \times (b \times c) = (a \times b) \times c$)
9. $\times$ has an identity, called 1 (i.e. for every a in the field, $a \times 1 = a$)
10. $\times$ has inverses, except for 0 (i.e. for every $a \neq 0$ in the field, there is some b in the field, with $a \times b = 1$. We write a^{-1} for this special b.)
11. $0 \neq 1$ (i.e. the additive and multiplicative inverses are not the same element)
12. $\times$ is distributive over $+$ (i.e. for every a, b, c in the field, $a \times (b + c) = (a \times b) + (a \times c)$)

Ordered: A field is ordered if it possesses a relation $\leq$, that satisfies the following five axioms:

1. $\leq$ is antisymmetric (i.e. for every a, b in the field, if $a \leq b$ and $b \leq a$, then $a = b$)
2. $\leq$ is transitive (i.e. for every a, b, c in the field, if $a \leq b$ and $b \leq c$, then $a \leq c$)
3. $\leq$ is a total order (i.e. for every a, b in the field, either $a \leq b$ or $b \leq a$ (or both, if $a = b$))
4. $\leq$ respects $+$ (i.e. for every a, b, c in the field, if $a \leq b$, then $a + c \leq b + c$)
5. $\leq$ respects $\times$ (i.e. for every a, b in the field, if $0 \leq a$ and $0 \leq b$, then $0 \leq a \times b$

Archim.: A field is Archimedean if every element is finite. That is, for every a in the field, there is some integer n (defined as $1 + 1 + \cdots + 1$, with n terms) with $a < n$.

Dense: A field is dense if for every a, b in the field with $b > 0$, there is some c in the field with $b > |c - a| > 0$. (in particular, c can't equal a)

Richard Dedekind 1831-1916

Lived and worked in Braunschweig, in central Germany.

"Of all the aids which the human mind has yet created to simplify its life - that is, to simplify the work in which thinking consists - none is so momentous and so inseparably bound up with the minds most inward nature as the concept of number. Arithmetic, whose sole object is this concept, is already a science of immeasurable breadth, and there can be no doubt that there are absolutely no limits to its further development; and the domain of its application is equally immeasurable, for every thinking person, even if he does not clearly realize it, is a person of numbers, an arithmetician."

Using $\mathbb{Q}$, we define the numbers $\prec L|R\succ = \prec L(x)|R(x)\succ$, where L and R are each subsets of $\mathbb{Q}$ (in particular, all the elements of L and R are fractions). Notationally, if $x = \prec L|R\succ$ we use x^L to mean the typical element of L; in other words, a statement like "$x^L < 2$" means "for all $a \in L, a < 2$". Similarly, we use x^R for the typical element of R. We have $x = \prec L|R\succ = \prec x^L|x^R\succ$.

There are many possible such numbers, because there are many possible subsets L, R. We take all the numbers $x = \prec L|R\succ$ that satisfy the following three axioms, and call them the 'real' numbers $\mathbb{R}$:

1. L, R are each nonempty; further, each has neither greatest element nor least element.
2. $x^L < x^R$ (every element of L is less than every element of R)
3. $|\mathbb{Q} \setminus (L \cup R)| \leq 1$ That is, together L and R include all fractions with at most one exception. Further:
   (a) If $L \cup R = \mathbb{Q}$, we call x 'irrational'.
   (b) If one fraction, a, is missing from $L \cup R$, then we must have $x^L < a < x^R$ (a is bigger than every element of L and smaller than every element of R). We call x 'rational', and say that $x = a$.

We can now define arithmetic with these 'real' numbers. Let $x = \prec x^L|x^R\succ, y = \prec y^L|y^R\succ$ be two such.

Ordering	We say $x \leq y$ when $x^L < y^R$. (i.e. every element of the left set of x is smaller than every element of the right set of y)
Equality	We say $x = y$ when $x \leq y$ and $y \leq x$.
Addition	$x + y = \prec x^L + y^L | x^R + y^R \succ$
Negation	$-x = \prec -x^R | -x^L \succ$
Absolute Value	$|x|$ equals x (if $x \geq 0$), or $-x$ (otherwise).
Multiplication	If $x, y \geq 0$, then we define $R(x \times y) = \{x^R \times y^R\}$, and $L(x \times y) = L(0) \cup \{x^L \times y^L | x^L \geq 0, y^L \geq 0\}$.

Our planned pace is: Basic properties (ex. 1-5) on Monday, Intermediate properties (ex. 6-9) on Wednesday, Advanced properties (ex. 10-12) on Monday.

For the first two exercises, set $L_1 = \{a|a \in \mathbb{Q}, a < 2\}, L_2 = \{a|a \in \mathbb{Q}, a < 3\}, L_3 = \{a|a \in \mathbb{Q}, a^2 < 2$ or $a < 0\}, R_1 = \{a|a \in \mathbb{Q}, a > 2\}, R_2 = \{a|a \in \mathbb{Q}, a > 3\}, R_3 = \{a|a \in \mathbb{Q}, a^2 > 2$ and $a > 0\}$, and set $x = \prec L_1|R_1\succ, y = \prec L_2|R_2\succ, z = \prec L_3|R_3\succ$.

1. Calculate $x + y, x \times x, x \times y$, using the above definitions. Prove that x, y, and $x + y$ are in $\mathbb{R}$ (they satisfy the three above axioms for $\mathbb{R}$). Prove that $0 \leq x$ and $x \leq x + y$.
2. Calculate $x + z, z \times z, x \times z$, using the above definitions. Determine, with proof, which of these three numbers are rational.
3. Prove that for any real $x = \prec L|R\succ$, that L, R each must contain infinitely many elements.
4. For any $x = \prec L|R\succ, y = \prec L'|R'\succ$, prove that $x \leq y$ iff $L' \supseteq L$. ($\leq$ defined above)
5. For any $x = \prec L|R\succ, y = \prec L'|R'\succ$, prove that $x = y$ iff $L' = L$. ($=$ defined above)
6. Prove that $\mathbb{R}$ is Archimedean.
7. Prove that $\mathbb{R}$ is dense.
8. Prove that the sum of two rationals is rational, and the product of two positive rationals is rational. Find two irrationals whose sum is rational. (remember to use the above definitions throughout)
9. Prove that $\mathbb{R}$ is ordered (five things to check, the first one is free since $=$ is defined this way).
10. Prove the first five field axioms for $\mathbb{R}$.
11. Extend the above definition of multiplication to all $x, y \in \mathbb{R}$. (hint: cases and absolute value)
12. Prove the last seven field axioms for $\mathbb{R}$. You need not prove every case of axioms 7, 8, and 12; you may prove the special case of $a, b, c \geq 0$. However, there will be multiple cases in axioms 6, 9, and 10.