MATH601 Spring 2008
Handout 16: Partisan Games (Hackenbush)
Unit 7: Games

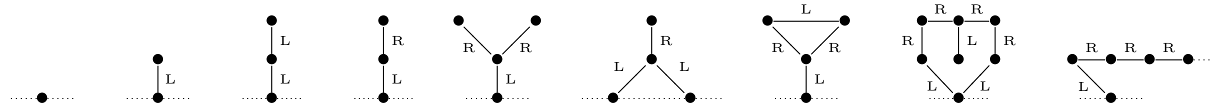
We define "Partisan" games. This is a special category of games, between two players (L,R), which satisfy:

1. Finite (game must end),
2. Perfect information (no secrecy),
3. Deterministic (no randomness involved),
4. Partisan (L's moves are different from R's moves),
5. No ties allowed, and
6. A player unable to move loses

can all be solved, in principle, with surreal numbers. Partisan games include chess, checkers, go, mancala, Connect Four, Othello, Tic-Tac-Toe, and many others. Note: some games need to be modified slightly by eliminating ties using rule 6 above – e.g. modified tic-tac-toe can always be won by skillful play of the first player because when the board is full it's the second player's move.

For partisan games, we say the *value* of a position is a surreal number x, where $L(x)$ is the set of positions after each of L's possible moves, and $R(x)$ is the set of positions after each of R's possible moves. A positive surreal is an advantage for L of that many moves; a negative surreal is an advantage to R. The surreal zero $\prec|\succ$ represents a loss for whomever must move.

We will play a game called Hackenbush. A position is a bush (a clump of bushes is considered a bush) growing out of the ground. The bush consists of dots (vertices) and edges. Each edge is labeled with L or R (you may use two colors if you prefer). A turn consists of removing one edge with your label, after which any part of the bush no longer connected to the ground disappears.



(i) (ii) (iii) (iv) (v) (vi) (vii) (viii) (ix)

Bush (i) above has value 0, since neither player can move. From bush (ii), L has one move, leading to (i), while R has no moves. Hence bush (ii) has value $\prec 0|\succ = 1$. From bush (iii) above, L can either move to bush (ii) or (i), while R has no moves. Hence bush (iii) has value $\prec 0,1|\succ = \prec 1|\succ = 2$.

Exercises:

1. Calculate the value of each of the bushes (iv)-(ix) above. (ix) has an infinite chain of R's.
   Notes: (viii) will take a long time. (ix) has an infinite bush, but is still a finite game.
2. If bush X has value x, and bush Y has value y, prove that putting both bushes on the ground next to each other will have value $x + y$ (using surreal addition).
3. Demonstrate that bush (iv) really has value $1/2$ by combining two of those with a bush of value -1, and showing that whoever goes first in this game loses.
4. Find a bush whose value is $9/64$.
5. Suppose we now allow some edges to have no labels; these may be removed by either player. This game is called Tri-Hackenbush, and is not partisan. We may still calculate value as before; however the result might not be a surreal number. Find a Tri-Hackenbush position whose value is not a number.
6. Make a list of various games; for each determine which (if any) of the above properties fail.
   e.g. Poker does not have perfect information, is not deterministic, and ties are allowed.
7. Back to regular Hackenbush. Let bush X have value x. Define bush Y by replacing each L label with R (and vice versa). Prove that bush Y has value $-x$.
8. Let bush X have value x. Prove that x is a number (i.e. $x^L < x < x^R$). You may assume there are only finitely many vertices.