MATH601 Spring 2008
Handout 15: Multiplication
Unit 6: Surreal Numbers

Multiplication is defined in the following, messy, way. However, it satisfies nice properties.

$x \cdot y = \prec x^L \cdot y + x \cdot y^L - x^L \cdot y^L, x^R \cdot y + x \cdot y^R - x^R \cdot y^R \mid x^L \cdot y + x \cdot y^R - x^L \cdot y^R, x^R \cdot y + x \cdot y^L - x^R \cdot y^L \succ$

Thm 1. For all surreal x, y, z, all of the following are satisfied: if $x \geq 0$ and $y \geq 0$, then $x \cdot y \geq 0$; $x \cdot 0 = 0$; $x \cdot 1 = x$; $x \cdot y = y \cdot x$; $(-x) \cdot y = x \cdot (-y) = -(x \cdot y)$; $(x \cdot y) \cdot z = x \cdot (y \cdot z)$; $(x + y) \cdot z = (x \cdot z) + (y \cdot z)$ (proved in the exercises)

Example 1: $2 \cdot 2 = \prec 1 | \succ \cdot \prec 1 | \succ = \prec 1 \cdot 2 + 2 \cdot 1 - 1 \cdot 1 | \succ$. Three of the four messy expressions are empty, since $R(x) = R(y) = \emptyset$. By Thm 1, $1 \cdot 2 = 2 = 2 \cdot 1$, and $1 \cdot 1 = 1$. We now compute $2 + 2 - 1 = 3$, so $2 \cdot 2 = \prec 3 | \succ = 4$.

Example 2: $1/2 \cdot 1/2 = \prec 0 | 1 \succ \cdot \prec 0 | 1 \succ = \prec 0 \cdot 1/2 + 1/2 \cdot 0 - 0 \cdot 0, 1 \cdot 1/2 + 1/2 \cdot 1 - 1 \cdot 1 | 0 \cdot 1/2 + 1/2 \cdot 1 - 0 \cdot 1, 1 \cdot 1/2 + 1/2 \cdot 1 - 1 \cdot 0 \succ$. By Thm 1, this equals $\prec 0 + 0 - 0, 1/2 + 1/2 - 1 | 0 + 1/2 - 0, 1/2 + 1/2 - 0 \succ = \, = \prec 0, 0 | 1/2, 1 \succ = \prec 0 | 1/2 \succ = 1/4$.

The following is quite useful, with a rather tricky proof that we will omit.

Thm 2. For all surreal a, b, x, y with $a \geq b$ and $x \geq y$, then $a \cdot x + b \cdot y \geq a \cdot y + b \cdot x$.

Exercises:

1. Compute $2 \cdot 3$, $5 \cdot 7$, $(-3) \cdot (-4)$, $3 \cdot 1/2$, $4 \cdot 1/4$, $1/\omega \cdot 1/\omega$, $\omega \cdot 1/\omega$.
2. For all surreal x, prove $x \cdot 0 = 0$.
3. For all surreal x, prove $x \cdot 1 = x$.
4. For all surreal x, y, prove $x \cdot y = y \cdot x$.
5. For all surreal x, y, prove $(-x) \cdot y = -(x \cdot y) = x \cdot (-y)$.
6. Use Theorem 2 to prove each of the following.
   (a) For all surreal x, y with $x \geq 0$ and $y \geq 0$, then $x \cdot y \geq 0$.
   (b) For all surreal x, y with $x \geq 0$ and $y \leq 0$, then $x \cdot y \leq 0$.
   (c) For all surreal x, y with $x \leq 0$ and $y \leq 0$, then $x \cdot y \geq 0$.
   (d) For all surreal x, y, z with $x \geq y$ and $z \geq 0$, then $x \cdot z \geq y \cdot z$.
   (e) For all surreal x, y, z with $x \geq y$ and $z \leq 0$, then $x \cdot z \leq y \cdot z$.
7. For all surreal x, y, z prove $(x \cdot y) \cdot z = x \cdot (y \cdot z)$.