MATH601 Spring 2008
Handout 14: Addition and Surreal Induction
Unit 6: Surreal Numbers

Addition is defined as: $x + y = \prec x^L + y, x + y^L | x^R + y, x + y^R \succ$

Compute $0 + 0 = \prec 0^L + 0, 0 + 0^L | 0^R + 0, 0 + 0^R \succ$. Since $R(0) = L(0) = \emptyset$, $0 + 0 = \prec | \succ = 0$.

Compute $0 + 1 = \prec 0^L + 1, 0 + 1^L | 0^R + 1, 0 + 1^R \succ = \prec 0 + 1^L | \succ = \prec 0 + 0 | \succ = \prec 0 | \succ = 1$.

Compute $1 + 0 = \prec 1^L + 0, 1 + 0^L | 1^R + 0, 1 + 0^R \succ = \prec 1^L + 0 | \succ = \prec 0 + 0 | \succ = \prec 0 | \succ = 1$.

Compute $1 + 1 = \prec 1^L + 1, 1 + 1^L | 1^R + 1, 1 + 1^R \succ = \prec 0 + 1, 0 + 1 | \succ = \prec 1, 1 | \succ = \prec 1 | \succ = 2$.

Compute $-1 + 1/2 = \prec -1^L + 1/2, -1 + 1/2^L | -1^R + 1/2, -1 + 1/2^R \succ = \prec -1 + 0 | 0 + 1/2, -1 + 1 \succ$:

$-1 + 0 = \prec -1^L + 0, -1 + 0^L | -1^R + 0, -1 + 0^R \succ = \prec | 0 + 0 \succ = \prec | 0 \succ = -1$

$0 + 1/2 = \prec 0^L + 1/2, 0 + 1/2^L | 0^R + 1/2, 0 + 1/2^R \succ = \prec 0 + 0 | 0 + 1 \succ = \prec 0 | 1 \succ = 1/2$

$-1 + 1 = \prec -1^L + 1, -1 + 1^L | -1^R + 1, -1 + 1^R \succ = \prec -1 + 0 | 0 + 1 \succ = \prec -1 | 1 \succ = 0$

Note: $\prec -1 | 1 \succ = 0$ by the Seniority Principle.

Hence $-1 + 1/2 = \prec -1 | 1/2, 0 \succ$. Because $1/2 > 0$, we apply exercise 8 from the previous handout. Hence $-1 + 1/2 = \prec -1 | 0 \succ = -1/2$.

We now introduce *Surreal Induction*. To prove some property $P(x, y)$, we assume as inductive hypothesis every possible combination where x, y are replaced by any of their left or right sets. That is, we assume as inductive hypothesis any or all of $P(x^L, y), P(x^R, y), P(x, y^L), P(x, y^R)$, $P(x^L, y^L), P((x^L)^L, (y^L)^R), \ldots$. In addition, because surreal numbers are built from nothing, there will often be no base case required.

Thm 1. For every surreal x, prove $x \geq x$ without the Seniority Principle.

Proof. First note that $x \geq x$ is equivalent to $x^R > x > x^L$. Let $a \in R(x), b \in L(x)$. We must prove $a > x > b$. By surreal induction, $a \geq a$, so $a > x$. By surreal induction, $b \geq b$, so $x > b$. □

Thm 2. For every surreal x, y, prove that $x + y = y + x$.

Proof. $x + y = \prec x^L + y, x + y^L | x^R + y, x + y^R \succ$. Applying the surreal inductive hypothesis on all four parts, we have $x + y = \prec y + x^L, y^L + x | y + x^R, y^R + x \succ = \prec y^L + x, y + x^L | y^R + x, y + x^R \succ = y + x$. □

Thm 3. For every surreal x, y, z with $x \geq y$ and $y \geq z$, prove that $x \geq z$. Further, equality holds (i.e. $x = z$) iff both $x = y$ and $y = z$.

Proof. Suppose that $x \geq y \geq z$. Because $x \geq y$, $x^R > y$; hence by surreal induction $x^R > z$. Because $y \geq z$, $y > z^L$; hence by surreal induction $x > z^L$. Combining these we get $x \geq z$.
Suppose now that $x > y$. Case (i): there is some $a \in R(y)$ with $x \geq a$. Since $y \geq z$, $a > z$. Hence $x \geq a > z$, so by surreal induction $x > z$. Case (ii): there is some $b \in L(x)$ with $b \geq y$. Then $b \geq y \geq z$, so by surreal induction $b \geq z$. Hence $x > z$.
The remainder of this proof is left as an exercise. □

For every surreal x, we define $-x = \prec -x^R | -x^L \succ$.

Thm 4. For every surreal x, prove that $-(-x) = x$.

Proof. Set $y = -x, z = -y$. $L(z) = \{-a | a \in R(y)\} = \{-(-b) | b \in L(x)\} = L(x)$ by surreal induction. Similarly, $R(z) = \{-c | c \in L(y)\} = \{-(-d) | d \in R(x)\} = R(x)$ by surreal induction. □

Thm 5. For every surreal x, y, prove that $x \geq y$ iff $-y \geq -x$.

Proof. Suppose that $x \geq y$. Hence $x^R > y$ and $x > y^L$, hence $y \not\geq x^R$ and $y^L \not\geq x$, hence (by surreal induction) $-x^R \not\geq -y$ and $-x \not\geq -y^L$, hence $-y > -x^R$ and $-y^L > -x$, hence $-y \geq -x$. Suppose now that $-y \geq -x$. Hence $(-y)^R > -x$ and $-y > (-x)^L$, hence $-x \not\geq (-y)^R$ and $(-x)^L \not\geq -y$, hence (by surreal induction) $-(-y)^R \not\geq -(-x)$ and $-(-y) \not\geq -(-x)^L$, hence $-(-(y^L)) \not\geq -(-x)$ and $-(-y) \not\geq -(-(x^R))$, hence (by Thm. 4) $y^L \not\geq x$ and $y \not\geq x^R$, hence $x > y^L$ and $x^R > y$, hence $x \geq y$. □

Thm 6. For every surreal x, y, z, w with $x \geq y$ and $w \geq z$, prove that $x + w \geq y + z$. Further, equality holds ($x + w = y + z$) iff both $x = y$ and $w = z$.

Proof. Suppose that $x \geq y$ and $w \geq z$. Therefore $x^R > x \geq y$ and $w^R > w \geq z$; hence by surreal induction $x^R + w > y + z$ and $x + w^R > y + z$ and therefore $(x + w)^R > y + z$. Also, $x \geq y > y^L$ and $w \geq z > z^L$; hence by surreal induction $x + w > y^L + z$ and $x + w > y + z^L$ and therefore $x + w > (y + z)^L$. We have verified (i) and (ii) and hence $x + w \geq y + z$.
Suppose now that $x > y$. Case (i): let $a \in R(y)$ with $x \geq a$. By surreal induction, $x + w \geq a + z$. But $a + z \in R(y + z)$, so $x + w > y + z$. Case (ii): let $b \in L(x)$ with $b \geq y$. By surreal induction, $b + w \geq y + z$. But $b + w \in L(x + w)$, so $x + w > y + z$.
The remainder of this proof is left as an exercise. □

Exercises:

1. Compute $-1 + 1$, $1/2 + 1/2, -1 + (-2), 1/4 + 3/4$.
2. Compute $\omega + 1$ and show that it equals $\prec \omega | \succ$. Calculate $\prec 1, 2, \ldots | \omega \succ +1$ and show that it equals ω. Calculate $1/2\omega + 1/2\omega$ and show that it equals $1/\omega$.
3. For every surreal x, prove that $0 + x = x$.
4. For every surreal x, y, z prove that $(x + y) + z = x + (y + z)$.
5. Complete the proof of Thm. 3. You need to prove that $y > z$ implies $x > z$, and that $x = y$ and $y = z$ together imply that $x = z$.
6. Complete the proof of Thm 6. You need to prove that $w > z$ implies $x + w > y + z$, and that $x = y$ and $w = z$ together imply that $x + w = y + z$.
7. For every surreal x, y, prove that $-(x + y) = (-x) + (-y)$.
   Hint: Use surreal induction and the definitions of addition, negation repeatedly.
8. For every surreal x, prove that $x + (-x) = 0$.
   Hint: Prove $x + (-x) \geq 0$ and $0 \geq x + (-x)$. Use surreal induction and Thms 4,5.
9. For every surreal x, y, z, prove that $x \geq y$ iff $x + z \geq y + z$.
   Hint: For one direction, use Thms 1, 6. For the other direction, use Thms 1,6 and exercise 5.