MATH601 Spring 2008
Handout 12: Exponentiation and Cantor Normal Form
Unit 5: Ordinals

Recall that addition and multiplication of ordinals were defined recursively. We now define exponentiation recursively, as repeated multiplication. Let x, y be ordinals.

1. If $y = 0$, then $x^y = 1$. (in particular, $0^0 = 1$)
2. If y is a successor, then for some ordinal z, we have $y = z + 1$. We define $x^y = (x^z) \times x$.
3. If y is a limit ordinal, then $y = \lim_{z<y} z$. We then define $x^y = \lim_{z<y} (x^z)$.

Exercises:

1. Calculate $2^3, 1^\omega, 2^\omega, 2^{\omega+1}$. Exponentiation of natural numbers coincides with your intuition (except for $0^0 = 1$).
2. Prove that $0^x = 0$ and $1^x = 1$ for all nonzero ordinals x.
3. Prove that $x^y \times x^z = x^{y+z}$ for all ordinals x, y, z.
4. Show that $(\omega \times 2)^2 = \omega^2 \times 2 \neq \omega^2 \times 2^2$. Hence the law of exponents $(x \times y)^z = x^z \times y^z$ does not always hold for ordinals.

All ordinal numbers can be written uniquely in *Cantor Normal Form*, which can be considered a base-ω arithmetic. The CNF representation of ordinal y is $\omega^{x_1}n_1 + \omega^{x_2}n_2 + \cdots + \omega^{x_k}n_k$, where $k \in \mathbb{N}_0$ ($k = 0$ corresponds to $y = 0$), $n_i \in \mathbb{N}$, x_i are ordinals satisfying $x_1 > x_2 > \cdots > x_k \geq 0$.

CNF can be used to simplify ordinal arithmetic. Given nonzero y as above,
$y \times n = \omega^{x_1}n_1 n + \omega^{x_2}n_2 + \cdots + \omega^{x_k}n_k$ $(n > 0)$ — only the leading coefficient increases
$y \times \omega = \omega^{x_1+1}$ — everything except the first term (without coefficient) is lost
$y + \omega^{x_i}n = \omega^{x_1}n_1 + \omega^{x_2}n_2 + \cdots \omega^{x_i}(n_i + n)$ — all terms smaller than the added one are lost
Recall also that ordinals satisfy distributivity on the left: $a \times (b + c) = (a \times b) + (a \times c)$.

More Exercises:

5. Calculate the CNF for $(\omega^3 + \omega \times 4 + 7) + (\omega^2 \times 3 + \omega \times 2 + 5)$.
6. Calculate the CNF for $(\omega^2 \times 3 + \omega \times 2 + 5) + (\omega^3 + \omega \times 4 + 7)$.
7. Calculate the CNF for $(\omega^3 + \omega \times 4 + 7) \times (\omega^2 \times 3 + \omega \times 2 + 5)$.
8. Calculate the CNF for $(\omega^2 \times 3 + \omega \times 2 + 5) \times (\omega^3 + \omega \times 4 + 7)$.