MATH601 Spring 2008
Final Exam Solutions

1. Express the decimal number 12345 in senary (with the letter notation we used).

   $6^2 = 36, 6^3 = 216, 6^4 = 1296, 6^5 = 7776, 6^6 = 46676$
   12345 is between $1 \cdot 6^5$ and $2 \cdot 6^5$; hence the leading senary hand is A. $12345 - 1 \cdot 6^5 = 4569$, which is between $3 \cdot 6^4$ and $4 \cdot 6^4$; hence the next senary hand is C. $4569 - 3 \cdot 6^4 = 681$, which is between $3 \cdot 6^3$ and $4 \cdot 6^3$; hence the next senary hand is C. $681 - 3 \cdot 5^3 = 33$, which is less than even one 6^2; hence the next senary hand is Z. However, 33 is greater than $5 \cdot 6^1$; hence the next senary hand is E. $33 - 5 \cdot 6^1 = 3$, so the last senary hand is C. Putting it all together gives $ACCZEC$.

2. Prove or disprove that $\mathbb{R}$ is Archimedean.

   Let $x \in \mathbb{R}$. By the definition of the real numbers, there is some $a/b \in R(x)$, with $a/b \in \mathbb{Q}$. We prove below that $x < x^R$, so in particular $x < a/b$. But $a/b \leq a$, so $x < a$, an integer. [Alternatively, a/b is in $\mathbb{Q}$, which is known to be Archimedean, hence is less than some integer.] Hence $\mathbb{R}$ is Archimedean.

   Proof that $x < x^R$: Let $y \in R(x)$. Suppose that $x \geq y$. Because $R(x)$ has no least element, then there is some $z < y$, with $z \in R(x)$. But z, y are both rationals, so $z \in L(y)$. Hence, by definition of $x \geq y$, we must have $z < z$, which is impossible.

3. Use the hyperreal definition of derivative to calculate $f'(x)$, for $f(x) = \sin(2x)$.

   We recall Thm 2. from Handout 6: for any nonzero infinitesimal ε, $st\left(\frac{\sin(\varepsilon)}{\varepsilon}\right) = 1, st\left(\frac{\cos(\varepsilon)-1}{\varepsilon}\right) = 0$.

   $$f'(x) = st\left(\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right) = st\left(\frac{\sin(2x+2\varepsilon)-\sin(2x)}{\varepsilon}\right) = st\left(\frac{\sin(2x)\cos(2\varepsilon)+\cos(2x)\sin(2\varepsilon)-\sin(2x)}{\varepsilon}\right) =$$
   $$st\left(\sin(2x)\frac{\cos(2\varepsilon)-1}{\varepsilon} + \cos(2x)\frac{\sin(2\varepsilon)}{\varepsilon}\right) = st\left(2\sin(2x)\frac{\cos(2\varepsilon)-1}{2\varepsilon} + 2\cos(2x)\frac{\sin(2\varepsilon)}{2\varepsilon}\right) =$$
   $$2\sin(2x) st\left(\frac{cos(2\varepsilon)-1}{2\varepsilon}\right) + 2\cos(2x) st\left(\frac{\sin(2\varepsilon)}{2\varepsilon}\right) = 2\sin(2x)\cdot 0 + 2\cos(2x)\cdot 1 = 2\cos(2x).$$

4. Find sets S, T such that $S = T \cup \{x\}$ for some single element $x \notin T$, and yet $|S| = |T|$.

   Many solutions are possible, so long as both sets are infinite. For example, let S be the set of nonnegative integers, and T the set of positive integers. $S = T \cup \{0\}$. They are equicardinal because of the function $f(x) = x + 1$. We show that this is a bijection from S to T: Let $t \in T$, then $t - 1 \in S$, and $f(t-1) = t$. Hence f is onto. Suppose that $f(s) = f(s')$. Then $s + 1 = s' + 1$, and hence $s = s'$. So f is one-to-one.

5. State the usual definition of the axiom of choice, and give two other equivalent statements.

   Given any set of nonempty sets, we can choose exactly one element from each of the nonempty sets.
   Any two cardinals are comparable, i.e. for any sets S, T either $|S| \leq |T|$ or $|T| \leq |S|$.
   For any two sets (at least one infinite), then $|S| \times |T| = |S| + |T| = \max\{|S|, |T|\}$.
   Every set can be well-ordered.
   Every vector space has a basis.
   Zorn's lemma: Every poset where every chain has an upper bound has a maximal element.
   Tychonoff's theorem: Every product of compact topological spaces is compact.

6. Prove that $x^y \times x^z = x^{y+z}$ for all ordinals x, y, z.

   Proof by transfinite induction on z. If $z = 0$, then $x^0 = 1$ by the definition of exponentiation; $x^y \times 1 = x^y$ by the lemma below; $x^{y+0} = x^y$ by the definition of addition. Hence the two sides are equal.

   If z is a successor, then $z = w+1$. Then $x^y \times x^z = x^y \times x^w \times x^1 = x^{y+w} \times x^1 = x^{y+w} \times (x^0 \times x) = x^{y+w} \times (1 \times x) = x^{y+w} \times x = x^{y+w+1} = x^{y+z}$ using (in order) definition of exponentiation, inductive hypothesis, definition of exponentiation, definition of exponentiation, the lemma below, definition of exponentiation, and associativity of addition.

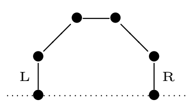
   If z is a limit, then $z = \lim_{w<z} w$. $x^y \times x^z = \lim_{w<z} x^y \times x^w = \lim_{w<z} x^{y+w} = x^{y+\lim_{w<z} w} = x^{y+z}$, using (in order) definition of exponentiation, inductive hypothesis, sloppy moving of limits, and assumption on z.

   Lemma: $1 \times x = x \times 1 = x$. This was proved on Exam 5.

7. Prove or disprove that $\mathbb{S}$, the set of surreal numbers, is dense.

   Let a, b be any surreal numbers with $b > 0$. By Thm 1. on Handout 14, $a \geq a$. By Thm 6. on Handout 14, $a + b > a + 0$. By an exercise from Handout 14, $a + 0 = a$. Hence $a + b > a$. Set $c = \prec a | a + b \succ$, which is a surreal number by the above calculation. By the Seniority Principle, $a < c < a+b$. Applying Thms 1,6 from Handout 14 again, we find $a + (-a) < c - a < (a+b) - a$. By an exercise from Handout 14, $a - a = 0$, so this simplifies to $0 < c - a < b$. Since $c > a$ we have $|c - a| = c - a$ so in fact $0 < |c - a| < b$. Hence $\mathbb{S}$ is dense.

8. Label the three unlabeled edges of this position of Hackenbush so that the resulting bush has value $1/2$:



   For convenience, we use $\cdot$ to indicate the ground; the problem is then $\cdot L???R\cdot$. There are eight possible ways to label the ?'s; however ones like $\cdot LLLLR\cdot$ where the L's and R's are separate have integer values, e.g. $\cdot LLLLR\cdot = \cdot LLLL + \cdot R = 4 - 1 = 3$. Hence by considering symmetry there are only essentially two patterns to check $\cdot LRLLR\cdot$, and $\cdot LLRLR\cdot$. The first one works. We first calculate a few basic bushes:

   $\cdot LR = \prec \cdot | \cdot L \succ = \prec 0 | 1 \succ = 1/2$ and similarly $\cdot RL = -1/2$
   $\cdot LRR = \prec \cdot | \cdot L, \cdot LR \succ = \prec 0 | 1/2, 1 \succ = 1/4$ and similarly $\cdot RLL = -1/4$
   $\cdot LRL = \prec \cdot, \cdot LR | \cdot L \succ = \prec 0, 1/2 | 1 \succ = 3/4$ and similarly $\cdot RLR = -3/4$
   $\cdot LRLL = \prec \cdot, \cdot LR, \cdot LRL | \cdot L \succ = \prec 0, 1/2, 3/4 | 1 \succ = 7/8$ and similarly $\cdot RLRR = -7/8$
   $\cdot LRRL = \prec \cdot, \cdot LRR | \cdot L, \cdot LR \succ = \prec 0, 1/4 | 1/2, 1 \succ = 3/8$ and similarly $\cdot RLLR = -3/8$.

   Putting it all together, we get $\cdot LRLLR\cdot = \prec \cdot RLLR, \cdot LR + \cdot RL, \cdot LRL + \cdot R | \cdot L + \cdot RLL, \cdot LRLL \succ =$ $= \prec -3/8, 1/2 - 1/2, 3/4 - 1 | 1 - 1/4, 7/8 \succ = \prec -3/8, -1/4, 0 | 3/4, 7/8 \succ = 1/2$ by the Seniority Principle.

   The other possible choice would be $\cdot LLRLR\cdot = \prec \cdot RLRL, \cdot RLR, \cdot R | \cdot LLRL, \cdot LL + \cdot RL \succ$
   $\cdot RLRL = \prec \cdot RLR, \cdot R | \cdot RL, \cdot \succ = \prec -3/4, -1 | -1/2, 0 \succ = -3/8$
   $\cdot LLR = \prec \cdot, \cdot L | \cdot LL \succ = \prec 0, 1 | 2 \succ = 3/2$
   $\cdot LLRL = \prec \cdot LLR, \cdot L, \cdot | \cdot LL \succ = \prec 3/2, 1, 0 | 2 \succ = 7/4$
   Hence $\cdot LLRLR\cdot = \prec -3/8, -3/4, -1 | 7/4, 2 - 1/2 \succ = 0$ by the Seniority Principle, so this doesn't work.