MATH601 Spring 2008
Exam 4 Solutions

1. Let R, S, T be sets. Suppose that $|S| < |T|$ (this means that $|S| \leq |T|$ but $|S| \neq |T|$). Suppose further that $|R| = |S|$. Prove that $|R| < |T|$. You may use only the definitions of $\leq$ and $=$, and the CSB theorem, but not any of the homework exercises.

   Because $|R| = |S|$, there is a bijection $f : R \to S$. Because $|S| \leq |T|$, there is an injection $g : S \to T$.

   First, we prove that $|R| \leq |T|$. Define a function $h : R \to T$ via $h(x) = g(f(x))$. We now prove that h is an injection from R to T. Suppose that $h(x) = h(x')$, i.e. $g(f(x)) = g(f(x'))$. Because g is an injection, $f(x) = f(x')$. Because f is an injection, $x = x'$.

   Now, we prove that $|R| \neq |T|$. Suppose otherwise; there would then be a bijection $H : R \to T$, and hence a bijection $H^{-1} : T \to R$. We now prove that $|T| \leq |S|$. Define a function $G : T \to S$ via $G(x) = f(H^{-1}(x))$. Suppose that $G(x) = G(x')$, i.e. $f(H^{-1}(x)) = f(H^{-1}(x'))$. Because f is an injection, $H^{-1}(x) = H^{-1}(x')$. Because H^{-1} is an injection, $x = x'$. Hence $|T| \leq |S|$; but also $|S| \leq |T|$. By the CSB theorem, $|S| = |T|$, which contradicts the assumption that $|S| \neq |T|$.

2. Let S be "the largest possible set", perhaps the set of all sets. Then surely $|S|$ is the largest possible cardinal. Find a larger cardinal, proving that there is no largest possible set.

   2^S is the power set of S, the set of all subsets of S. 2^S is a set whose cardinality is bigger than the cardinality of S. This is by Cantor's Theorem (Exercise 6 from handout 8), which states that $|2^S| > |S|$. Proving this was nice, but not necessary – I gave full credit for citing this as Cantor's Theorem or citing the homework exercises.

   This problem was essentially all-or-nothing, as the only method we have learned to generate a set of larger cardinality is by taking the power set. Solutions mentioning 2^S got most or all of the credit, while solutions that did not got very little credit.

3. Exam grades: 96, 90, 85, 75, 70, 66, 66, 58, 51