MATH601 Spring 2008
Exam 3 Solutions

1. For H any positive infinite hyperreal, compute $st\left(\sqrt{H^2+H+1}-H\right)$, or show that it does not exist.

   We focus on the argument: $\sqrt{H^2+H+1}-H = \left(\sqrt{H^2+H+1}-H\right)\frac{\sqrt{H^2+H+1}+H}{\sqrt{H^2+H+1}+H} = \frac{(H^2+H+1)-H^2}{\sqrt{H^2+H+1}+H} = \frac{H+1}{\sqrt{H^2+H+1}+H}\frac{1/H}{1/H} = \frac{1+1/H}{\sqrt{1+1/H+1/H^2}+1}$.

   We now take standard parts, and use Theorem 3 from Section 1.6 repeatedly to get: $st(\sqrt{H^2+H+1}-H) = st\left(\frac{1+1/H}{\sqrt{1+1/H+1/H^2}+1}\right) = \frac{st(1)+st(1/H)}{\sqrt{st(1)+st(1/H+1/H^2)}+st(1)} = \frac{1+0}{\sqrt{1+0}+1} = 1/2$

   (Note: By the "rules for infinitesimals", since H is infinite, so is H^2; hence $1/H$, $1/H^2$, and $1/H+1/H^2$ are infinitesimal. The standard part of any infinitesimal is 0.)

2. Prove the product rule: Given real functions $f(x), g(x)$, set $h(x)=f(x)g(x)$. Suppose that $f'(x), g'(x)$ both exist. Then $h'(x)$ exists, and equals $f'(x)g(x)+g'(x)f(x)$.

   Let ε be any nonzero hyperreal. We recall that $f'(x) = st(\frac{f(x+\varepsilon)-f(x)}{\varepsilon}), g'(x) = st(\frac{g(x+\varepsilon)-g(x)}{\varepsilon})$; since these are both defined, each of $f(x), f(x+\varepsilon), g(x), g(x+\varepsilon)$ are defined. Hence $h(x), h(x+\varepsilon)$ are defined.

   Using the definition of derivative, $h'(x) = st\left(\frac{h(x+\varepsilon)-h(x)}{\varepsilon}\right) = st\left(\frac{f(x+\varepsilon)g(x+\varepsilon)-f(x)g(x)}{\varepsilon}\right)$. The argument is: $\frac{f(x+\varepsilon)g(x+\varepsilon)-f(x)g(x)}{\varepsilon} = \frac{f(x+\varepsilon)g(x+\varepsilon)-f(x+\varepsilon)g(x)+f(x+\varepsilon)g(x)-f(x)g(x)}{\varepsilon} = f(x+\varepsilon)\frac{g(x+\varepsilon)-g(x)}{\varepsilon}+g(x)\frac{f(x+\varepsilon)-f(x)}{\varepsilon}$.

   Taking standard parts and using Theorem 3, we find $h'(x) = st\left(\frac{f(x+\varepsilon)g(x+\varepsilon)-f(x)g(x)}{\varepsilon}\right) = st\left(f(x+\varepsilon)\frac{g(x+\varepsilon)-g(x)}{\varepsilon}+g(x)\frac{f(x+\varepsilon)-f(x)}{\varepsilon}\right) = st(f(x+\varepsilon))g'(x)+st(g(x))f'(x) = f(x)g'(x)+g(x)f'(x)$.

   Note: $st(f(x+\varepsilon)) = f(x)$ by the increment theorem, and $st(g(x)) = g(x)$ since $g(x)$ is a real function. Finally, we note that this was independent of the choice of ε (so long as $\varepsilon \neq 0$), so $h'(x)$ is defined because $f'(x), g'(x)$ are.

3. Exam grades: 93, 89, 88, 85, 84, 84, 83, 79