1. For $H$ any positive infinite hyperreal, compute $st \left( \sqrt{H^2 + H + 1} - H \right)$, or show that it does not exist.

We focus on the argument: $\sqrt{H^2 + H + 1} - H = \left( \sqrt{H^2 + H + 1} - H \right) \frac{\sqrt{H^2 + H + 1} + H}{\sqrt{H^2 + H + 1} + H} = \frac{H + 1}{1 + \frac{1}{H^2 + H + 1}}$.

We now take standard parts, and use Theorem 3 from Section 1.6 repeatedly to get:

$$st(\sqrt{H^2 + H + 1} - H) = st \left( \frac{1 + 1/\epsilon}{\sqrt{1 + 1/\epsilon + 1/\epsilon^2 + 1}} \right) = \frac{st(1) + st(1/\epsilon)}{\sqrt{st(1) + st(1/\epsilon) + st(1)}} = 1/2$$

(Note: By the “rules for infinitesimals”, since $H$ is infinite, so is $H^2$; hence $1/\epsilon, 1/\epsilon^2$, and $1/\epsilon^2 + 1/\epsilon^2$ are infinitesimal. The standard part of any infinitesimal is 0.)

2. Prove the product rule: Given real functions $f(x), g(x)$, set $h(x) = f(x)g(x)$. Suppose that $f'(x), g'(x)$ both exist. Then $h'(x)$ exists, and equals $f'(x)g(x) + g'(x)f(x)$.

Let $\epsilon$ be any nonzero hyperreal. We recall that $f'(x) = st(\frac{f(x+\epsilon) - f(x)}{\epsilon}, g'(x) = st(\frac{g(x+\epsilon) - g(x)}{\epsilon}); since these are both defined, each of $f(x), f(x+\epsilon), g(x), g(x+\epsilon)$ are defined. Hence $h(x), h(x+\epsilon)$ are defined.

Using the definition of derivative, $h'(x) = st \left( \frac{h(x+\epsilon) - h(x)}{\epsilon} \right) = st \left( \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x)}{\epsilon} \right)$.

The argument is: $f(x+\epsilon)g(x+\epsilon) - f(x)g(x) = f(x+\epsilon)g(x+\epsilon) - f(x+\epsilon)g(x) + f(x+\epsilon)g(x) - f(x)g(x) = f(x+\epsilon)g(x+\epsilon) - f(x)g(x) + g(x)\frac{f(x+\epsilon) - f(x)}{\epsilon}$.

Taking standard parts and using Theorem 3, we find $h'(x) = st \left( \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x)}{\epsilon} \right) = st \left( \frac{f(x+\epsilon)g(x+\epsilon) - f(x)g(x)}{\epsilon} \right) = st(f(x+\epsilon))g'(x) + st(g(x))f'(x) = f(x)g'(x) + g(x)f'(x)$.

Note: $st(f(x+\epsilon)) = f(x)$ by the increment theorem, and $st(g(x)) = g(x)$ since $g(x)$ is a real function. Finally, we note that this was independent of the choice of $\epsilon$ (so long as $\epsilon \neq 0$), so $h'(x)$ is defined because $f'(x), g'(x)$ are.

3. Exam grades: 93, 89, 88, 85, 84, 84, 83, 79