1. (5-8 points) Let $\pi = (2 \ 4) \ (1 \ 5 \ 3)$. Calculate and simplify $\pi \circ \pi \circ \pi$.

$\pi \circ \pi \circ \pi = (2 \ 4) \ (2 \ 4) \ (1 \ 5 \ 3) \ (1 \ 5 \ 3) \ (1 \ 5 \ 3) = (2 \ 4)$.

2. Prove that $p(n)$ is equal to the number of partitions of $2n$ with no odd parts.

We create a function $f$ between partitions of $n$ and partitions of $2n$, that acts by doubling each part. The range of this function consists of exactly those partitions of $2n$ with all even parts, and on that range $f$ is invertible, hence it is a bijection.

3. Consider all partitions of 11. What is the maximal Durfee square? Give all partitions that yield this Durfee square.

We can fit a $3 \times 3$ Durfee square into the partitions, but not a $4 \times 4$, since $9 < 11 < 16$. The remaining two boxes in the Ferrers diagram can both be to the right of the square $(5 + 3 + 3, \ 4 + 4 + 3)$, or one can be to the right and one below $(4 + 3 + 3 + 1)$, or they can both be below the square $(3 + 3 + 3 + 2, \ 3 + 3 + 3 + 1 + 1)$. Together, these are five partitions.

4. Consider $\pi \in S_n$. Prove that, for any such choice of $\pi$, that $|\det A| = 1$, for matrix $A$ whose entries are given by $A_{i,j} = \begin{cases} 1 & \pi(i) = j \\ 0 & \text{otherwise.} \end{cases}$

Note that $A$ is a 0/1 matrix with exactly one 1 in each row and column. We will prove that all such matrices have determinant 1 or $-1$, by induction on the size $n$. If $n = 1$, then $A = [1]$, which has determinant 1. Otherwise, we expand $A$ on the first row, which has just one nonzero entry, so $\det A = 1(\pm 1) \det B$, where $B$ is a matrix obtained by deleting the first row and some column of $A$. Since $B$ is of the same form, $|\det B| = 1$ by the inductive hypothesis, so $|\det A| = 1$.

5. (5-12 points) Prove that $\left[\begin{array}{c} n \\ 2 \end{array}\right] \geq (n - 1)! + (n - 2)!$, for all integer $n \geq 2$.

We want to count permutations of $n$ with exactly two cycles. This includes those that have one cycle of length 1 and one cycle of length $n - 1$ (and lots of others). These are counted by Thm 6.9, as $\frac{n!}{(n-1)!} = n \cdot (n-2)! = (n-1) \cdot (n-2)! + 1 \cdot (n-2)! = (n-1)! + (n-2)!$.

In fact, $\left[\begin{array}{c} n \\ 2 \end{array}\right] = (n - 1)!(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n-1})$. 

MATH 579 Exam 6 Solutions