

## MATH 579 Spring 2010 Supplement: Recurrences

A recurrence is a sequence of numbers, defined by some positional relationship. This positional relationship is called a recurrence relation. That is, the  $n^{\text{th}}$  number is a function of the previous numbers. Some examples of recurrence relations are  $a_n = 2a_{n-1}$ ,  $b_n = b_{n-2} + 2$ ,  $c_n = c_{n-1} + c_{n-2}$  (Fibonacci numbers if  $c_1 = c_2 = 1$ ),  $d_{n+1} = n(d_{n-1} + d_n)$  (derangements if  $d_1 = 0, d_2 = 1$ ). To fully specify the sequence, ‘enough’ initial conditions are necessary. For example,  $\{a_n\}$  requires one initial condition (e.g.  $a_1 = 3$ ).  $\{b_n\}$  requires two;  $b_1 = 3$  is enough to specify all the odd terms in the sequence, but to specify the even terms we need  $b_2 = 4$ .

To solve a recurrence means to find a closed-form expression for the sequence, that does not depend on previous terms. Assuming you have psychic powers, the best way to solve recurrences is by guessing. A recurrence is completely specified by its initial conditions and recurrence. If you can guess the answer and show that your guess satisfies the recurrence and satisfies the initial conditions – this is enough to prove your answer.

**Example 1a:**  $a_1 = 1, a_n = 2a_{n-1}$  ( $n \geq 2$ )

Guess  $a_n = 2^{n-1}$ . Check that  $2^{1-1} = 1$ , so the initial condition is satisfied. Also,  $2^{n-1} = 2 \times 2^{(n-1)-1}$ , so the recurrence relation is satisfied.

Much as with differential equations, recurrences fall into many types, with many different strategies for solution. A *linear* recurrence relation of *order*  $k$  may be written as  $a_n = \star a_{n-1} + \star a_{n-2} + \cdots + \star a_{n-k} + \star$ , where each  $\star$  is some function of  $n$ . If each  $\star$  is, in fact, a constant, we say that the recurrence has *constant coefficients*. In this section, we will only consider linear recurrences. Further, we will assume that all the coefficients (except possibly the final  $\star$ ) are constants. If the final  $\star$  is identically zero (i.e.  $a_n = \star a_{n-1} + \star a_{n-2} + \cdots + \star a_{n-k}$ ) we call the relation *homogeneous*; otherwise we call it *nonhomogeneous*. In the above examples,  $a_n = 2a_{n-1}$  is first-order homogeneous with constant coefficients,  $b_n = b_{n-2} + 2$  is second-order nonhomogeneous with constant coefficients,  $c_n = c_{n-1} + c_{n-2}$  is second-order homogeneous with constant coefficients, and  $d_{n+1} = n(d_{n-1} + d_n)$  is second-order homogeneous with nonconstant coefficients.

### Homogeneous Linear Recurrence Relations with Constant Coefficients

We consider the recurrence relation  $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \cdots + c_{n-k}a_{n-k}$ . Because this is homogeneous, we may multiply a solution by any constant and it will be a solution. We may also add two solutions and get a solution. In short, the set of solutions forms a linear space. This space is of dimension  $k$ , because the relation is of order  $k$  and requires  $k$  initial conditions to fully specify the recurrence. Hence, to find the general solution, we may find  $k$  linearly independent solutions, and take all their linear combinations. Caution: be sure that the  $k$  specific solutions are linearly independent.

Let’s guess that  $a_n = x^n$  is a solution. We substitute into the recurrence to get  $x^n =$

$c_{n-1}x^{n-1} + c_{n-2}x^{n-2} + \dots + c_{n-k}x^{n-k}$ . Dividing by  $x^{n-k}$  gives us  $x^k = c_{n-1}x^{k-1} + c_{n-2}x^{k-2} + \dots + c_{n-k}$ . This is known as the *characteristic equation* of the recurrence relation. It is a polynomial of degree  $k$ , and therefore by the Fundamental Theorem of Algebra has  $k$  complex roots, counted by multiplicity.

If the  $k$  roots  $r_1, r_2, \dots, r_k$  are all distinct, then  $a_n = r_1^n, a_n = r_2^n, \dots, a_n = r_k^n$  are  $k$  linearly independent solutions, and therefore span the solution space. The general solution is therefore  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$ . The  $k$  initial conditions allow us to determine the unknown  $\alpha_1, \alpha_2, \dots, \alpha_k$  for a particular solution.

If, on the other hand, a root is repeated (i.e.  $r_1 = r_2$ ), then  $a_n = r_1^n, a_n = r_2^n, \dots, a_n = r_k^n$  are *NOT*  $k$  linearly independent solutions.  $\alpha_1 r_1^n + \alpha_2 r_2^n$  is a one-dimensional subspace, being equal to  $\alpha_1 r_1^n$  alone, because  $r_1 = r_2$ . Fortunately, if a root is repeated, we have available to us additional solutions, that are linearly independent. If root  $r_1$  has multiplicity 4, then  $r_1^n, nr_1^n, n^2 r_1^n, n^3 r_1^n$  are four linearly independent solutions (this fact will not be proved). In this manner we again get  $k$  linearly independent solutions, and therefore the general solution via linear combinations.

**Example 1b:**  $a_1 = 1, a_n = 2a_{n-1}$  ( $n \geq 2$ )

This has characteristic equation  $x = 2$ ; hence the general solution is  $a_n = \alpha 2^n$ . Substituting  $n = 1$  and using the initial conditions, we have  $1 = a_1 = \alpha 2^1$ . We solve to find  $\alpha = 1/2$ ; hence the specific solution is  $a_n = (1/2)2^n = 2^{n-1}$ .

**Example 2:**  $a_1 = a_2 = 1, a_n = a_{n-1} + a_{n-2}$  ( $n \geq 3$ ) (Fibonacci numbers)

This has characteristic equation  $x^2 = x + 1$ , which has roots (using the quadratic formula)  $r_1 = (1 + \sqrt{5})/2$  and  $r_2 = (1 - \sqrt{5})/2$ . Hence the general solution is  $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ . We have two initial conditions:  $1 = a_1 = \alpha_1 r_1 + \alpha_2 r_2, 1 = a_2 = \alpha_1 r_1^2 + \alpha_2 r_2^2$ . This is a  $2 \times 2$  linear system in the unknowns  $\alpha_1, \alpha_2$ , with solution  $\alpha_1 = \frac{1}{\sqrt{5}}, \alpha_2 = \frac{-1}{\sqrt{5}}$ . Hence the specific solution is  $a_n = (r_1^n - r_2^n)/\sqrt{5}$ .

**Example 3:**  $a_0 = a_2 = 1, a_1 = 0, a_3 = 2, a_n = -a_{n-1} + 3a_{n-2} + 5a_{n-3} + 2a_{n-4}$  ( $n \geq 4$ )

This has characteristic equation  $x^4 + x^3 - 3x^2 - 5x - 2 = 0$ . We find the roots by guessing small integers (the rational root theorem helps too); if we successfully guess a root  $r$ , we divide by  $x - r$  using long division and continue. In this manner, we find roots  $-1$  (multiplicity 3), and 2. Hence, the general solution is  $a_n = \alpha_1(-1)^n + \alpha_2 n(-1)^n + \alpha_3 n^2(-1)^n + \alpha_4 2^n$ . We now apply our initial conditions to get:

$$(n = 0) : 1 = a_0 = \alpha_1 + \alpha_4 \quad (n = 1) : 0 = a_1 = \alpha_1(-1) + \alpha_2(-1) + \alpha_3(-1) + \alpha_4 2$$

$$(n = 2) : 1 = a_2 = \alpha_1 + \alpha_2 2 + \alpha_3 4 + \alpha_4 4 \quad (n = 3) : 2 = a_3 = \alpha_1(-1) + \alpha_2(-3) + \alpha_3(-9) + \alpha_4 8$$

This is a  $4 \times 4$  linear system, with solution  $\alpha_1 = 7/9, \alpha_2 = -3/9, \alpha_3 = 0, \alpha_4 = 2/9$ . Therefore, the specific solution is  $a_n = (7/9)(-1)^n - (3n/9)(-1)^n + (2/9)2^n$ .

**Example 4 (Gambler's ruin):** A gambler repeatedly plays a game against a casino, until one of them runs out of money. Each time the gambler has probability  $p$  of winning \$1, and probability  $q = 1 - p$  of losing \$1. The gambler starts with  $n$  dollars, and the casino with  $m - n$  dollars (there are  $m$  total dollars to be won). What is the probability that the gambler will run out of money before the casino?

Let  $a_n$  denote the desired probability, that the gambler is successful starting with  $n$  dollars. For the gambler to win, either (1) gambler wins first bet, and then is successful starting with  $n + 1$  dollars, or (2) gambler loses first bet, and then is successful starting with  $n - 1$  dollars. Therefore, this sequence satisfies the recurrence relation  $a_n = pa_{n+1} + qa_{n-1}$  ( $0 < n < m$ ), with boundary conditions  $a_0 = 1, a_m = 0$ . This has characteristic equation  $px^2 - x + q = 0$ , with roots  $r_1 = 1, r_2 = q/p$ . Hence the problem breaks into two cases, depending on whether  $p = q$  or not.

( $p \neq q$ ): The general solution is  $a_n = \alpha 1^n + \beta r_2^n = \alpha + \beta r_2^n$ . We apply the boundary conditions, to get ( $n = 0$ ) :  $1 = a_0 = \alpha + \beta$ , ( $n = m$ ) :  $0 = a_m = \alpha + \beta r_2^m$ . This has solution  $\alpha = -r_2^m / (1 - r_2^m), \beta = 1 / (1 - r_2^m)$ . Hence, the specific solution is  $(-r_2^m + r_2^n) / (1 - r_2^m) = 1 - \frac{1 - r_2^n}{1 - r_2^m}$ .

( $p = q = 1/2$ ): The general solution is  $a_n = \alpha 1^n + \beta n 1^n = \alpha + \beta n$ . We apply the boundary conditions, to get ( $n = 0$ ) :  $1 = a_0 = \alpha$ , ( $n = m$ ) :  $0 = a_m = \alpha + \beta m$ . This has solution  $\alpha = 1, \beta = -1/m$ . Hence, the specific solution is  $1 - (n/m)$ .

## Nonhomogeneous Linear Recurrence Relations

We want to solve the nonhomogeneous recurrence relation  $a_n = c_{n-1}a_{n-1} + c_{n-2}a_{n-2} + \dots + c_{n-k}a_{n-k} + b(n)$ , where  $b(n)$  is a function of  $n$ . The technique to find the general solution is in two parts. First, drop the  $b(n)$  term and find the general solution to the homogeneous recurrence relation. Then, find any single solution to the nonhomogeneous recurrence (under any initial/boundary conditions). The general solution to the nonhomogeneous recurrence is the sum of these two – a  $k$ -dimensional term from the homogeneous part, and a single term with no constants from the nonhomogeneous part.

Finding a particular solution is, at times, an art form. The only good way to find them is to guess and check – guess a particular solution, and see if it fits the nonhomogeneous relation. If  $b(n)$  is a polynomial, it's a good idea to try guessing a polynomial of the same degree; however, if the homogeneous solution has overlap with this, then increase the degree of your guess. If  $b(n)$  is an exponential, it's a good idea to try a multiple of the same exponential.

**Example 5:**  $a_0 = 2, a_n = 2a_{n-1} + 3^n$  ( $n \geq 1$ )

Homogeneous version:  $a_n = 2a_{n-1}$ , which has characteristic equation  $x = 2$  and general solution  $\alpha 2^n$ .

Nonhomogeneous version: Let's guess  $\beta 3^n$ . Plugging into the relation, we get  $\beta 3^n =$

$2\beta 3^{n-1} + 3^n$ . We divide both sides by  $3^{n-1}$  to get  $3\beta = 2\beta + 3$ ; hence  $\beta = 3$ . Thus  $3^{n+1}$  is a specific solution to the original, nonhomogeneous, recurrence.

Putting them together, we find the general solution to the nonhomogeneous recurrence is  $a_n = \alpha 2^n + 3^{n+1}$ . We now consider the initial condition,  $(n = 0) : 2 = a_0 = \alpha 2^0 + 3^1$ . This has solution  $\alpha = -1$ , and so the specific solution is  $a_n = 3^{n+1} - 2^n$ .

**Example 6:**  $a_0 = a_1 = 1, a_n = 2a_{n-1} - a_{n-2} + 5^n$  ( $n \geq 2$ )

Homogeneous version:  $a_n = 2a_{n-1} - a_{n-2}$ , which has characteristic equation  $x^2 - 2x + 1 = 0$ . This has a double root of 1, hence has general solution  $\alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$ .

Nonhomogeneous version: Let's guess  $\beta 5^n$ . Plugging into the relation, we get  $\beta 5^n = 2\beta 5^{n-1} - \beta 5^{n-2} + 5^n$ . We divide both sides by  $5^{n-2}$  to get  $25\beta = 10\beta - \beta + 25$ . This has solution  $\beta = 25/16$ , so a nonhomogeneous solution is  $(25/16)5^n = 5^{n+2}/16$ .

Putting them together, we find the general solution to the nonhomogeneous recurrence is  $a_n = \alpha_1 + \alpha_2 n + 5^{n+2}/16$ . Considering the initial conditions,  $(n = 0) : 1 = a_0 = \alpha_1 + 25/16$ ,  $(n = 1) : 1 = a_1 = \alpha_1 + \alpha_2 + 125/16$ . This has solution  $\alpha_1 = -9/16, \alpha_2 = -132/16$ , and so the specific solution is  $a_n = (-9 - 132n + 5^{n+2})/16$ .

**Example 7:**  $a_0 = 2, a_n = 3a_{n-1} - 4n$  ( $n \geq 1$ )

Homogeneous version:  $a_n = 3a_{n-1}$ , which has characteristic equation  $x = 3$  and general solution  $\alpha 3^n$ .

Nonhomogeneous version: We guess a solution of  $\beta_1 n + \beta_0$ . Plugging into the nonhomogeneous equation, we get  $(\beta_1 n + \beta_0) = 3(\beta_1(n-1) + \beta_0) - 4n$ . Simplifying, we get  $0 = (2\beta_1 - 4)n + (-3\beta_1 + 2\beta_0)$ . If a polynomial equals zero, then each coefficient must equal zero; hence  $0 = 2\beta_1 - 4$  and  $0 = -3\beta_1 + 2\beta_0$ . We solve this system to get  $\beta_1 = 2, \beta_0 = 3$ . Hence  $2n + 3$  is a solution to the nonhomogeneous recurrence.

Putting them together, we find the general solution to the nonhomogeneous recurrence is  $a_n = \alpha 3^n + 2n + 3$ . With our initial condition, we have  $(n = 0) : 2 = a_0 = \alpha 3^0 + 3$ , so  $\alpha = -1$ . So the specific solution is  $a_n = -3^n + 2n + 3$ .

**Example 8 (Tower of Hanoi):** We have three pegs and  $n$  disks of different sizes. The disks all start on one peg arranged in order of size, and we must move them to another. We move one disk at a time, and may never put a larger disk onto a smaller. How many moves does it take?

Let  $a_n$  represent the answer. We see that  $a_1 = 1$ . To move the biggest disk from peg 1 to peg 2, all the smaller disks must be in a single stack, on peg 3. Therefore, the solution must contain three steps: First, move the  $n - 1$  smaller disks from peg 1 to peg 3, then move the largest disk from peg 1 to peg 2, then move the  $n - 1$  smaller disks back onto the largest disk from peg 3 to peg 2. Hence,  $a_n = a_{n-1} + 1 + a_{n-1} = 2a_{n-1} + 1$ .

The homogeneous recurrence is again  $a_n = 2a_{n-1}$  with general solution  $\alpha 2^n$ . To find a specific

solution to the nonhomogeneous recurrence, consider a constant (0-th degree) polynomial in  $n$ , say  $\beta$ . Plugging into the nonhomogeneous equation, we get  $\beta = 2\beta + 1$ ; we solve this to get  $\beta = -1$ . Hence the general solution to the nonhomogeneous relation is  $a_n = \alpha 2^n - 1$ . Our initial conditions tell us  $1 = a_1 = \alpha 2^1 - 1$ ; hence  $\alpha = 1$  and our specific solution is  $a_n = 2^n - 1$ .

**Example 9 (Gambler's ruin revisited):** Consider the gambler of example 4. What is the expected number of games played until either the gambler or casino is ruined?

Let  $a_n$  denote the desired answer (when the gambler starts with  $\$n$ ). If the gambler wins, then the expected number of games is one more than the expected number of games, had the gambler started with  $\$(n + 1)$ . If the gambler loses, then the expected number of games is one more than the expected number of games, had the gambler started with  $\$(n - 1)$ . Hence we get the relation  $a_n = p(a_{n+1} + 1) + q(a_{n-1} + 1)$  ( $0 < n < m$ ). We have boundary conditions  $0 = a_0 = a_m$ , and may rewrite the relation as  $pa_{n+1} = a_n - qa_{n-1} - 1$ . The homogeneous recurrence has the familiar characteristic equation  $px^2 - x + q = 0$ ; once again the problem splits into cases based on whether  $q = p$ .

( $p \neq q$ ): The homogeneous general solution is  $\alpha + \beta r_2^n$  (recall that  $r_2 = q/p$ ). If we try to guess a 0-th degree polynomial solution to the nonhomogeneous recurrence, we will find no luck (try it and see). The reason is that all 0-th degree polynomials are already solutions of the homogeneous recurrence, and so none of them could ever solve the nonhomogeneous recurrence.

Instead let's try a first-degree polynomial  $c_1 n + c_0$ . We plug into the nonhomogeneous equation to get  $p(c_1(n + 1) + c_0) = c_1 n + c_0 - q(c_1(n - 1) + c_0) - 1$ . We collect terms to get  $n(pc_1 - c_1 + qc_1) + (pc_1 + pc_0 - c_0 - qc_1 + qc_0 + 1) = 0$ . The first coefficient is zero already, and the second coefficient simplifies to  $(p - q)c_1 + 1 = 0$ ; hence  $c_1 = -1/(p - q)$ , and we may as well take  $c_0 = 0$  although the choice is arbitrary (in fact, we could have known this since all constants are part of the homogeneous solution). Therefore, the general nonhomogeneous solution is  $a_n = \alpha + \beta r_2^n - n/(p - q)$ . For the particular solution, we take  $0 = a_0 = \alpha + \beta, 0 = a_m = \alpha + \beta r_2^m - m/(p - q)$ . This has solution  $\beta = \frac{m}{(1-2p)(1-r_2^m)}, \alpha = -\beta$ . We plug these into the general solution, to find  $a_n = \left(n - m \frac{1-r_2^n}{1-r_2^m}\right)/(1 - 2p)$ .

( $p = q = 1/2$ ): The homogeneous general solution is  $\alpha + \beta n$ . We won't get very far trying low-degree polynomials, since they are all part of the homogeneous solution. So, let's try  $cn^2$ . We plug into the nonhomogeneous equation to get  $pc(n + 1)^2 = cn^2 - qc(n - 1)^2 - 1$ . We rewrite to get  $n^2(pc - c + qc) + n(2pc - 2qc) + (pc + qc + 1) = 0$ . Since  $p = q = 1/2$ , the first two coefficients are zero already, and the last is zero when  $c = -1$ . Hence the general nonhomogeneous solution is  $a_n = \alpha + \beta n - n^2$ . For the specific solution, we take  $0 = a_0 = \alpha, 0 = a_m = \alpha + \beta m - m^2$ . This has solution  $\alpha = 0, \beta = m$ . Hence, the specific solution is  $a_n = mn - n^2 = n(m - n)$ .

## Exercises

Solve the following recurrence relations.

1.  $a_0 = a_1 = 2, a_n = -2a_{n-1} - a_{n-2} (n \geq 2)$
2.  $a_0 = 0, a_1 = 1, a_n = 4a_{n-2} (n \geq 2)$
3.  $a_0 = 2, a_1 = -4, a_2 = 26, a_n = a_{n-1} + 8a_{n-2} - 12a_{n-3} (n \geq 3)$
4.  $a_0 = a_1 = a_2 = 0, a_n = 9a_{n-1} - 27a_{n-2} + 27a_{n-3} (n \geq 3)$
5.  $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + 3 (n \geq 2)$
6.  $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + n (n \geq 2)$
7.  $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + e^n (n \geq 2)$
8. What is the maximum number of regions we can divide the plane into, using  $n$  lines?
9. Let  $a_n$  be the number of  $n$ -digit nonnegative integers in which no three consecutive digits are the same. Justify that  $a_{n+2} = 9a_{n+1} + 9a_n$ , then find  $a_n$ .
10. Let  $a_n$  be the number of ways to color the squares of a  $1 \times n$  chessboard using the colors red, white, and blue, so that no two red squares are adjacent.
11. Let  $a_n$  be the number of ways to color the squares of a  $1 \times n$  chessboard using the colors red, white, and blue, so that no red square is adjacent to a white square.
12. Let  $a_n$  be the number of ways to color the squares of a  $1 \times n$  chessboard using the colors red, white, and blue, so that the specific sequence red-white-blue does not occur.
13. Let  $a_n$  be the number of ways to climb a flight of  $n$  stairs, when each of your steps may move you one, two, or three steps higher.
14. Codewords (strings) from the alphabet  $\{0, 1, 2, 3\}$  are called *legitimate* if they have an even number of 0's. How many legitimate codewords are there, of length  $k$ ?