MATH 579 Exam 5 Solutions

Part I: Prove that \( p(n) \leq \frac{p(n-1)+p(n+1)}{2} \), for \( n \in \mathbb{N} \).

A bit of algebra shows that the statement is equivalent to \( q(n+1) \geq q(n) \), for the function \( q(n) = p(n) - p(n-1) \). Let’s call a “nice” partition one where each part is at least 2. Thm 5.20 in the text states that \( q(n) \) counts nice partitions. We now establish a bijection between nice partitions of \( n \) and certain nice partitions of \( n + 1 \), namely the ones that have their largest part strictly bigger than the second-largest part. Given a nice partition of \( n \), we add 1 to the largest part. This gives a nice partition of \( n + 1 \), which is a bijection between the two sets in question. Hence \( q(n+1) \geq q(n) \).

NOTE: It isn’t enough to add 1 to an arbitrary part of a nice partition of \( n \); that is not 1-1.

Part II:

1. Find a formula for \( S(n, 2) \), for \( n \geq 2 \).

The number of surjective functions from \([n]\) to \([2]\) is \(2!S(n, 2)\). There are \(2^n\) functions altogether; however two are not surjective: the one that sends everything to 1, and the one that sends everything to 2. Solving \(2^n - 2 = 2S(n, 2)\) we get \(S(n, 2) = 2^{n-1} - 1\).

2. Find a formula for \( S(n, 3) \), for \( n \geq 3 \).

The number of surjective functions from \([n]\) to \([3]\) is \(3!S(n, 3)\). There are \(3^n\) functions altogether; however three send everything to just one place, and \(\binom{3}{2}(2^n-2)\) send everything to two places (applying the previous problem). Hence \(3^n - 3(2^n - 2) - 3 = 6S(n, 3)\); solving, we get \(S(n, 3) = 0.5(3^{n-1} - 2^n + 1)\).

3. Find the number of compositions of 25 into 5 odd parts.

By subtracting one from each part, we get a bijection between compositions of 25 into 5 odd parts, and weak compositions of 20 (=25-5) into 5 even parts. By dividing each part in half, we get a bijection between weak compositions of 20 into 5 even parts, and weak compositions of 10 into 5 parts. For
this we have a formula, namely \( \binom{14}{10} = 1001 \).

4. Prove that \( p_k(n) \leq (n - k + 1)^{k-1} \), for \( 1 \leq k \leq n \).

We give a process that will yield various partitions with \( k \) parts, among them all partitions of \( n \) into \( k \) parts. For each part, we select from \([1, n - k + 1]\), and we do this \( k - 1 \) times.

For the last part, there is at most one possible choice to make the sum \( n \); if possible, we take it, otherwise it doesn’t matter what we take. This process has \( (n - k + 1)^{k-1} \) outcomes.

We now show that every possible partition of \( n \) into \( k \) parts occurs, by showing that each such partition must have each part at most \( n - k + 1 \). If not, then some part must be greater than this, but the other \( k - 1 \) parts have sum at least \( k - 1 \), so together the sum would be greater than \( n \).

5. Prove that \( B(n) \geq \binom{n}{2} \), for \( n \geq 0 \).

**SOLUTION 1:** Thm 5.12 states: \( B(n+1) = \sum_i \binom{n}{i} B(i) \). We first prove the lemma that \( B(n) \geq n \). We proceed by strong induction on \( n \); for \( n = 0 \) the claim is \( 0 \geq 0 \), which is true.

Now \( B(n + 1) = \sum_i \binom{n}{i} B(i) \geq \sum_{i\in[0,n]} 1 = n + 1 \).

Now we use the lemma to prove our result. \( B(n + 1) = \sum_i \binom{n}{i} B(i) \geq \sum_{i\in[0,n]} i = \frac{n(n+1)}{2} = \binom{n+1}{2} \), as desired.

**SOLUTION 2:** Induction on \( n \); we need extra base cases because we need \( n \geq 3 \) in our induction: \( B(0) = 1 \geq 0 = \binom{0}{2} \), \( B(1) = 1 \geq 0 = \binom{1}{2} \), \( B(2) = 2 \geq 1 = \binom{2}{2} \), and \( B(3) = 3 \geq 3 = \binom{3}{2} \). By Thm. 5.12, \( B(n + 1) = \sum_i \binom{n}{i} B(i) \geq \sum_i \binom{n}{i} \left( \frac{i}{2} \right) \geq \frac{n(n+1)}{2} = \binom{n+1}{2} \), where we used the inductive hypothesis at the beginning and \( n \geq 3 \) at the end (to prove \( 2n - 2 \geq n + 1 \)).

**SOLUTION 3:** Bell numbers are defined as \( B(n) = \sum_k S(n, k) \geq S(n, n-1) = \binom{n}{2} \) (for \( n \geq 2 \)). For \( n = 0, 1 \), we use \( B(0) = 1 \geq 0 = \binom{0}{2} \), \( B(1) = 1 \geq 0 = \binom{1}{2} \).

Exam grades: High score=104, Median score=66 (ouch!), Low score=52