MATH 579 Exam 5 Solutions

1. Calculate the number of compositions of 14 into an even number of even parts.

Partitions of 14 into even parts are bijective with partitions of 7 into integer parts, by dividing each part by 2. We don’t want all such though, we insist on an even number of parts, namely 2 or 4 or 6. Applying Cor. 5.3 thrice, the answer is \( \binom{14}{3} + \binom{14}{4} + \binom{14}{6} = 6 + 20 + 6 = 32. \)

2. For all \( n \in \mathbb{N}, \) determine \( S(n, n - 2). \)

There are two types of set partitions of \( [n] \) into \( n - 2 \) parts. First, there is the type that has one triple and \( n - 3 \) singletons. There are \( \binom{n}{3} \) such. Second, there is the type that has two doubles and \( n - 4 \) singletons. If the doubles were different, there would be \( \binom{n}{2}(\binom{n-2}{2}) \) such; however, they are not, so in fact there are \( \frac{1}{2!}\binom{n}{2}(\binom{n-2}{2}) \) such. Putting it together, we get \( \binom{n}{3} + \frac{1}{2!}\binom{n}{2}(\binom{n-2}{2}) \). Note that this works even for \( n = 1, 2, \) where everything is 0.

3. Calculate \( S(8, 3). \)

Using the helpful but not necessary formula \( S(n, 2) = 2^{n-1} - 1, \) together with Thm 5.8, we get \( S(3, 3) = 1, S(4, 3) = S(3, 2) + 3S(3, 3) = (2^2 - 1) + 3 = 6, S(5, 3) = S(4, 2) + 3S(4, 3) = (2^4 - 1) + 3(6) = 25, S(6, 3) = S(5, 2) + 3S(5, 3) = (2^4 - 1) + 3(25) = 90, S(7, 3) = S(6, 2) + 3S(6, 3) = (2^4 - 1) + 3(90) = 301, S(8, 3) = S(7, 2) + 3S(7, 3) = (2^6 - 1) + 3(301) = 966. \)

4. Let \( a_n \) denote the number of compositions of \( n \) where each part is larger than 1. Find a formula relating \( a_n, a_{n-1}, a_{n-2}. \)

We divide such compositions into two types: A: those that have first term equal to 2, B: those that have first term greater than 2. Type A are bijective with compositions counted by \( a_{n-2}, \) as seen by removing that first term. Type B are bijective with compositions counted by \( a_{n-1}, \) as seen by subtracting one from the first term. Hence \( a_n = a_{n-1} + a_{n-2}. \) Note that \( a_2 = 1, a_3 = 1, \) so in fact these are the Fibonacci numbers in disguise.

5. Let \( k \) range from 0 to \( n \), prove that \( \sum \binom{n}{k} S(k, l) S(n - k, m) = S(n, l + m) \binom{l+m}{l}. \)

We count partitions of \( [n] \) into \( l \) nonempty “red” parts, and \( m \) nonempty “blue” parts. One way to do this is to first partition \( [n] \) into \( l + m \) nonempty parts, and then paint \( l \) of them red (the rest are blue). The RHS counts this way. Another way is to first choose \( k \) elements that will be in a red part; we then partition them into nonempty parts in \( S(k, l) \) ways. The remaining \( n - k \) elements will be in a blue part; we partition them in \( S(n - k, m) \) ways. The LHS counts this approach.

6. Let \( p \) be prime, prove that \( B(p) \equiv 2 \pmod{p}. \) Equivalently, prove that \( p \) divides \( B(p) - 2. \)

Consider the function \( f \) on partitions of \( [p] \) that acts by permuting the numbers within the parts as \( 1 \to 2 \to 3 \to \cdots \to p \to 1. \) For example, for \( p = 3, f \) acts as \( \begin{cases} \{1, 2, \{3\} \to \{2, 3\} \{1\} \to \{1, 3\} \{2\} \to \{1, 2\} \{3\} \end{cases}. \) Call two partitions ‘equivalent’ if some number of applications of \( f \) will map one onto the other. \( f \) leaves exactly two partitions alone: \( \{1\} \{2, \ldots, p\} \) and \( \{1, 2, \ldots, p\}. \) All other partitions are equivalent to exactly \( p \) partitions [special case of Lagrange’s theorem]; hence \( B(p) \) is two plus some multiple of \( p. \)

Note 1: Since the cycle of partitions that \( f \) induces all have the same number of parts, this also proves that \( p \mid S(p, k), \) for \( p \) prime and \( 1 < k < p. \)

Note 2: \( p \) must be prime for this to hold. For example, for \( p = 4, \) the cycle \( \begin{cases} \{1, 2\} \{3, 4\} \to \{2, 3\} \{1, 4\} \to \{1, 2\} \{3, 4\} \end{cases} \) only has two partitions, not \( p. \) And indeed \( B(4) = 15, \) which is not congruent to 2 modulo 4.

Note 3: This result is a special case of Touchard’s Congruence: \( B_n + B_{n+1} \equiv B_{n+p} \pmod{p}. \) This problem corresponds to \( n = 0; \) the general result can be proved in a similar way.

Exam results: High score=88, Median score=70, Low score=53 (before any extra credit)