MATH 579 Exam 4 Solutions

1. For all \( n \in \mathbb{N} \), prove that \( \sum_k \binom{n}{k} \binom{2n}{n-k} = \binom{3n}{n} \).

First solution: Apply Thm. 4.7 in the text, with \( m = 2n \), \( k = n \), \( i = k \).

Second solution: In a class with \( n \) women and \( 2n \) men, we choose \( n \) students. There are \( \binom{3n}{n} \) ways to do this, or we could instead choose \( k \) women and \( n-k \) men, for every possible \( k \in [0, n] \). The sum counts exactly this, because for \( k \) outside this range the summand is zero.

This is a direct application of Thm 4.7. It is also very similar to Problem 33, and to Problem 3 from the exam two years ago.

2. A northeastern lattice path is a path consisting of \((1,0)\) and \((0,1)\) steps. How many such paths are there from \((0,0)\) to \((10,10)\) that do not pass through \((1,1)\)?

We count northeast lattice paths in a rectangle of size \( n \times k \). There must be \( k \) steps north, and \( n \) steps east. Hence the northeast lattice paths are bijective with rearrangements of the word \(NN \cdots NE \cdots E\), which is counted by the multinomial coefficient \( \binom{n+k}{n,k} = \binom{n+k}{k} \).

First solution: There are \( \binom{10+10}{10} \) \( \text{NE} \) lattice paths altogether. We count how many pass through \((1,1)\) and subtract. This is a product of the number of paths from \((0,0)\) to \((1,1)\) and the number of paths from \((1,1)\) to \((10,10)\). Hence the answer is \( \binom{20}{10} - \binom{2}{1} \binom{18}{9} = 184756 - 97240 = 87516 \).

Second solution: Desired paths must start \( EE \) or \( NN \). The former then has a northeast path from \((2,0)\) to \((10,10)\), of which there are \( \binom{8+10}{10} = 43758 \). The latter has a northeast path from \((0,2)\) to \((10,10)\), of which there are \( \binom{10+8}{8} = 43758 \). Altogether there are \( 43758 + 43758 = 87516 \).

This is very similar to Problems 19, 23, 24, 31, 32, 50.

3. Which monomial term(s) of \((x+y+z)^{16}\) has the largest coefficient? What is that coefficient?

If \( x^a y^b z^c \) is in the expansion, it must have \( a+b+c = 16 \). Its coefficient is \( \frac{16!}{a!b!c!} \).

To maximize the coefficient we must minimize the denominator. Note that if \( a > b+1 \), then \( (a-1)+(b+1)+c = a+b+c = 6 \), and \( \frac{(a-1)(b+1)c!}{a!b!c!} = \frac{b+1}{a} < 1 \), so \( (a-1)(b+1)c! < a!b!c! \) and hence \( a!b!c! \) could not have been minimal. Hence, by symmetry, \( a, b, c \) must all be either equal or differ by 1. Hence two of them are 5 and one is 6. There are therefore three terms, each with coefficient \( \binom{16}{5,5,6} : 2018016x^5y^5z^6, 2018016x^5y^6z^5, 2018016x^6y^5z^5 \).

This is very similar to Problems 10,11,12,44,45.

4. For all \( n \in \mathbb{N} \), calculate \( \sum_{k \text{ odd}} \binom{n}{k} 3^k \).

By Newton’s binomial theorem, we have \( (1+3)^n = \sum_k \binom{n}{k} 3^k \). Also, \( (1-3)^n = \sum_{k \text{ odd}} \binom{n}{k} (-3)^k \).
\[
\sum_k \binom{n}{k} (-3)^k = \sum_k \binom{n}{k} 3^k (-1)^k. \quad \text{Hence } 4^n - (-2)^n = \sum_k \binom{n}{k} 3^k (1 - (-1)^k) = \sum_{k, \text{odd}} \binom{n}{k} 3^k 2. \quad \text{Hence } \sum_{k, \text{odd}} \binom{n}{k} 3^k = \frac{4^n - (-2)^n}{2} = 2^{n-1} (2^n - (-1)^n).
\]

This is a direct application of theorems proved in class. It is also very similar to Problems 39, 40.

5. For all \(k \in \mathbb{Z}\), prove that \(\binom{n-1}{k-1} \binom{n}{k+1} = \binom{n-1}{k-1} \binom{n}{k+1} \binom{n}{k-1}\).

First, if \(k \leq 0\), then both sides are zero since \(\binom{n-1}{k-1} = \binom{n}{k-1} = 0\). Otherwise, \(k \geq 1\), and we have \(LHS = \binom{n-1}{k-1} \binom{n}{k+1} \binom{n}{k-1}\) and \(RHS = \binom{n-1}{k-1} \binom{n}{k+1} \binom{n}{k-1}\). Note that both denominators are \((k-1)!k!(k+1)!\) so it suffices to prove that \(A = B\), for \(A = \binom{n-1}{k-1} \binom{n}{k+1} \binom{n}{k-1}\) and \(B = \binom{n-1}{k} \binom{n+1}{k+1} \binom{n}{k-1}\). These have gcd \(C = \binom{n-1}{k-1} \binom{n}{k+1} \binom{n}{k-1}\). We have \(A = C(n - (k + 1) + 2)(n - (k + 1) + 1) = C(n-k+1)(n-k), B = C(n-1-k+1)(n+1-(k+1)+1) = C(n-k)(n-k+1). \quad \text{Hence } A = B, \text{ which proves the theorem.}

This is solved directly from the definition of binomial coefficients; there are many problems that explore this idea.

6. For \(m, n \in \mathbb{N}\), prove that \(\frac{1}{m^n!} = \lim_{n \to \infty} \frac{m+n}{n} n^{-m}\).

We have \(\frac{1}{m^n!} = \lim_{n \to \infty} \frac{(m+n)!}{m^n n!} n^{-m} = \frac{1}{m^n} \lim_{n \to \infty} \frac{(m+n)!}{m^n n!} = \frac{1}{m^n} \lim_{n \to \infty} \frac{m! (m+n) \cdots (n+1)}{n!} = \frac{1}{m^n} \lim_{n \to \infty} \frac{(1+m/n) \cdots (1+1/n)}{n!} \). This looks bad until you realize that there are \(m\) terms in the product, and \(m\) is fixed as \(n \to \infty\). Hence this equals \(\frac{1}{m^n} \lim_{n \to \infty} (1 + \frac{m}{n})(1 + \frac{m}{n}) \cdots (1 + \frac{1}{n}) = \frac{1}{m^n} \times 1 \times \cdots \times 1 = \frac{1}{m^n} \). This theorem was discovered and proved by the great mathematician Leonard Euler, at the age of 22.

This is solved directly from the definition of binomial coefficients; there are many problems that explore this idea.

Exam results: High score = 84, Median score = 68, Low score = 52 (before any extra credit)