Math 579 Exam 8 Solutions

1. Carefully define the following three terms: formal power series, (ordinary) generating function, partial fractions.

   A formal power series is just $\sum_{k\geq 0} a_k x^k$, where the a_k are a sequence of constants; it need not converge. An ordinary generating function is a formal power series, associated to the sequence $\{a_k\}$. Partial fractions is a reduction technique to rewrite a rational function as a sum of other rational functions whose denominators are irreducible.

2. Find a generating function that can be used to count how many a, b, c, d there are that solve $a+b+c+d=n$ and satisfy:
   (1) a, b, c, d are nonnegative integers,
   (2) a is 2,3, or 7, and
   (3) b, c are multiples of 3.

   $a : (x^2+x^3+x^7) \quad b, c : (1+x^3+x^6+x^9+\cdots) = 1/(1-x^3)$
   $d : (1+x+x^2+x^3+\cdots) = 1/(1-x)$. Multiplying these together, we get $\frac{x^2+x^3+x^7}{(1-x^3)^2(1-x)}$.

3. $A(x) = \frac{x^5+3x-2}{(1+x)^5}$ is the generating function for a sequence a_n. Find a closed form for a_n (for $n \geq 5$ is sufficient).

   $A(x) = (x^5+3x-2)\sum_{n\geq 0}\binom{4+n}{n}(-1)^n x^n = (x^5+3x-2)\sum_{n\geq 0}\binom{4+n}{4}(-1)^n x^n =$
   $\sum_{n\geq 0}\binom{4+n}{4}(-1)^n x^{n+5} + \sum_{n\geq 0} 3\binom{4+n}{4}(-1)^n x^{n+1} + \sum_{n\geq 0} -2\binom{4+n}{4}(-1)^n x^n =$
   $\sum_{n\geq 5}\binom{n-1}{4}(-1)^{n-5} x^n + \sum_{n\geq 1} 3\binom{3+n}{4}(-1)^{n-1} x^n + \sum_{n\geq 0} -2\binom{4+n}{4}(-1)^n x^n =$
   some low-degree terms $+ \sum_{n\geq 5}\left(\binom{n-1}{4}(-1)^{n-5} + 3\binom{3+n}{4}(-1)^{n-1} - 2\binom{4+n}{4}(-1)^n\right)x^n$
   Hence, for $n \geq 5$, $a_n = \left(\binom{n-1}{4} + 3\binom{n+3}{4} + 2\binom{n+4}{4}\right)(-1)^{n+1}$.

4. $a_0 = 0, a_{k+1} = 2a_k + 2^k$. Using generating functions, find a closed form for a_k.

   Multiply both sides of the recurrence by x^{k+1} and sum over all $k \geq 0$ to get $\sum_{k\geq 0} a_{k+1}x^{k+1} = 2x\sum_{k\geq 0} a_k x^k + x\sum_{k\geq 0}(2x)^k$. Set $A(x) = \sum_{k\geq 0} a_k x^k$; observe that $\sum_{k\geq 1} a_k x^k = A(x) - a_0 = A(x)$. Hence, we conclude that $A(x) = 2xA(x) + x/(1-2x)$. We rearrange and solve for $A(x) = x/(1-2x)^2 = x\sum_{k\geq 0}(k+1)(2x)^k = \sum_{k\geq 0}(k+1)2^k x^{k+1} = \sum_{k\geq 1} k2^{k-1}x^k$. Hence $a_k = k2^{k-1}$.

5. $a_0 = 1, a_1 = 5, a_n = a_{n-1} + 2a_{n-2}$ $(n \geq 2)$. Using g.f., find a closed form for a_n.

   Multiply both sides of the recurrence by x^n and sum over all $n \geq 2$ to get
   $\sum_{n\geq 2} a_n x^n = x\sum_{n\geq 2} a_{n-1}x^{n-1} + 2x^2\sum_{n\geq 2} a_{n-2}x^{n-2}$.
   Set $A(x) = 1 + 5x + \sum_{n\geq 2} a_n x^n$; the above equation then becomes $(A(x)-1-5x) = x(A(x)-1) + 2x^2A(x)$. We rewrite to get $-1-4x = A(x)(2x^2+x-1)$, which we solve for $A(x) = \frac{-4x-1}{2x^2+x-1} = \frac{-4x-1}{(2x-1)(x+1)}$. Using the technique of partial fractions, we have $-4x-1 = B(x+1) + C(2x-1)$, which has solution $B=-2, C=-1$. Hence $A(x) = -2/(2x-1) - 1/(x+1) = 2/(1-2x) - 1/(1+x) = 2\sum_{n\geq 0} 2^n x^n - \sum_{n\geq 0}(-1)^n x^n = \sum_{n\geq 0}(2^{n+1} + (-1)^{n+1})x^n$. Hence $a_n = 2^{n+1} + (-1)^{n+1}$.

Part II. Consider two four-sided (tetrahedral) dice, with their sides numbered $1, 2, 3, 4$ in the ordinary way. Consider the experiment of throwing them together and taking the total. This can have outcomes between 2 and 8, inclusive.

(a) Determine the probability of each of these outcomes.

(b) Determine all other ways (if any) of numbering the sides of two such dice, so that these same outcomes are possible and have the same probabilities as with the ordinary dice.

The ordinary dice each have generating function $f(x) = x + x^2 + x^3 + x^4$, hence the experiment has outcomes given by $f(x)^2 = (x + x^2 + x^3 + x^4)^2 = x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + x^8$. Plugging in $x = 1$ shows that there are 16 possible, equally likely, outcomes, with probabilities given by:

2	3	4	5	6	7	8
1/16	2/16	3/16	4/16	3/16	2/16	1/16

We factor $f(x) = x(x+1)(x^2+1)$; hence $f(x)^2 = x^2(x+1)^2(x^2+1)^2$. Suppose we had two other four-sided dice, numbered differently, as $A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4}$ (a_1, a_2, a_3, a_4 are the positive integer numberings on the first die), and $B(x) = x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4}$. To have the same probabilities, we must have $A(x)B(x) = f(x)^2 = x^2(x+1)^2(x^2+1)^2$. By the unique factorization of polynomials, we know that there must be nonnegative integers $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$ such that $A(x) = x^{\alpha_1}(x+1)^{\alpha_2}(x^2+1)^{\alpha_3}$ and $B(x) = x^{\beta_1}(x+1)^{\beta_2}(x^2+1)^{\beta_3}$. Further, $\alpha_1 + \beta_1 = \alpha_2 + \beta_2 = \alpha_3 + \beta_3 = 2$.

Because $A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4}$, we must have $A(0) = 0$. However, if $\alpha_1 = 0$, $A(0) = (0+1)^{\alpha_2}(0^2+1)^{\alpha_3} = 1$. Hence $\alpha_1 \geq 1$; but, similarly, $\beta_1 \geq 1$. Therefore $\alpha_1 = \beta_1 = 1$.

Because $A(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4}$, we must have $A(1) = 4$. We have $4 = A(1) = 1^{\alpha_1}(1+1)^{\alpha_2}(1^2+1)^{\alpha_3} = 2^{\alpha_2+\alpha_3}$, so $\alpha_2 + \alpha_3 = 2$. The ordinary numbering corresponds to $\alpha_2 = \alpha_3 = 1$; however, it is also possible to have $\alpha_2 = 2, \alpha_3 = 0$. This makes $A(x) = x(x+1)^2(x^2+1)^0 = x + 2x^2 + x^3$, $B(x) = x(x+1)^0(x^2+1)^2 = x + 2x^3 + x^5$. (If $\alpha_2 = 0, \alpha_3 = 2$ that is the same solution, reversing A, B).

There is therefore exactly one other way to number the faces of the two dice: $\{1, 2, 2, 3\}$ and $\{1, 3, 3, 5\}$ so that the sums have the same probabilities as with the ordinary numbering.

NOTE: This solution assumes that we insist on positive integers to number the sides of the dice. If we permit zero, then we get several new numberings: $\{A(x) = x^2(x+1)(x^2+1), B(x) = (x+1)(x^2+1)\}, \{A(x) = x^2(x+1)^2, B(x) = (x^2+1)^2\}, \{A(x) = (x+1)^2, B(x) = x^2(x^2+1)^2\}$. More generally, if we allow negative integers, then we may add k to each side of one die, and subtract k from each side of the other die; this corresponds to multiplying one polynomial by x^k, and dividing the other by x^k.

Exam statistics: Low grade=30(D-); Median grade=36(C); High grade=48(A)