Math 579 Exam 6 Solutions

1. Calculate $s(3,k)$ for all integer k. BONUS: What does this say about $(x)_3$?

   $s(3,k) = 0$ for $k \leq 0$ and $k > 3$; this leaves three cases. $s(3,1) = (-1)^2 c(3,1)$; since there are 2! permutations of [3] with one cycle [namely (123) and (132)], $s(3,1) = 2$. $s(3,2) = (-1)^1 c(3,2)$; since there are $\binom{3}{2}$ permutations of [3] with two cycles [namely (12), (13), and (23)], $s(3,2) = -3$. Finally, $s(3,3) = 1$. We may conclude from this that $(x)_3 = x^3 - 3x^2 + 2x$ (easily verified by multiplying out $(x)_3 = x(x-1)(x-2)$)

2. How many n-permutations contain $1, 2$, and 3 in three different cycles?

   We use the bijection between cycle notation and one-line notation. If the cycle notation is in my canonical form, then 1,2,3 are in different cycles precisely when 3 comes first, then 2, then 1. Looking at the one-line notation, this occurs exactly 1/6 of the time; hence the answer is $n!/6$. If you prefer the book's canonical notation, you'll first want a bijection between 1,2,3 and $n-2, n-1, n$.

3. Let $u(n)$ denote the number of n-permutations whose cube is the identity permutation. Find $u(6)$.

   SOLUTION 1: A permutation cubes to the identity when each of its cycles are of length 1 or 3. For $n = 6$, that means there are three types of permutations to count: (A) six 1-cycles, (B) one 3-cycle, three 1-cycles, (C) two 3-cycles. These may be counted with Theorem 6.1 in the text, or directly (as follows). There is only one of type (A), the identity. There are $\binom{6}{3}$ ways to pick three elements, and then 2! ways to build them into a cycle; hence there are 40 of type (B). Finally, there are $\binom{6}{3}/2$ ways to split the six elements into two halves (divide by two because picking 1,2,3 is the same as picking 4,5,6). Then, there are 2!2! ways to build these two halves each into a cycle; hence there are 40 of type (C). Putting it all together, $u(6) = 81$.

   SOLUTION 2: By techniques similar to problem 4, we begin by proving that $u(n) = u(n-1) + (n-1)(n-2)u(n-3)$. We use this lemma repeatedly to find $u(6) = u(5) + 20u(3) = (u(4) + 12u(2)) + 20u(3) = ((u(3) + 6u(1)) + 12u(2) + 20u(3) = 21u(3) + 12u(2) + 6u(1)$. Now, $u(1) = u(2) = 1$, since only the identity cubes to the identity with so few elements. $u(3) = 3$; in addition to the identity, we have (123) and (132). Hence $u(6) = 21 \times 3 + 12 + 6 = 81$.

4. Find a recursive formula for the number $t(n)$ of n-permutations whose fifth power is the identity permutation.

   A permutation has its fifth power the identity when each of its cycles are of length 1 or 5. Consider the first element, 1. If it is its own cycle, there are $t(n-1)$ permutations of the remaining elements into cycles of the desired size. Otherwise, let's put 1 into a cycle of length 5. In canonical form, 1 will be first in its cycle, so there are $(n-1)_4$ ways to build the rest of the cycle, and $t(n-5)$ permutations of the remaining elements into cycles of the desired size. Putting it together, $t(n) = t(n-1) + n(n-1)(n-2)(n-3)t(n-5)$.

5. Prove that $p^{n!}$ is the identity permutation, for every n-permutation p.

   Write p in cycle notation; its cycles will be of various lengths, but each at most n (since the sum of all the cycle lengths must be equal to n). We may calculate p^k by raising each cycle to the power k, and concatenating the result; the reason is that the cycles are disjoint from each other and therefore may commute. If a cycle is of length k, then when it is raised to the power k, the identity remains; each element goes exactly once around the cycle. Further, if a cycle is raised to a power mk, for any natural number m, the same still holds, because $c^{mk} = (c^k)^m = 1^m = 1$. Since the length of each cycle is at most n, this length must divide $n!$, hence each cycle raised to the power $n!$ must give the identity. Now we calculate $p^{n!}$ by raising each cycle to the power $n!$ (getting the identity) and concatenating the results.

   NOTE: This theorem is proved in Modern Algebra, as a consequence of Lagrange's Theorem for groups.

Part II. Let A be the set of 18 students enrolled in this course, and let $B = \{$3 Musketeers, Butterfinger, Hershey Bar, KitKat, Milky Way, Snickers$\}$, six candy bars. Design a problem yielding each of the following solutions. That is, for each of these values: (1) Carefully and completely describe a process involving A, B that has this many possible outcomes, and (2) Give at least two representative examples of what you're counting.

A) $\binom{18}{6}$: Six candy bars are given out, to distinct students. We care only about which students get candy, not about what type of candy. Example: {Burak, Dustin, Gary, Jeff, Jeremy, Joseph}.
B) $\left(\!\binom{18}{6}\!\right)$: Same as $\binom{18}{6}$, but a student may get more than one candy bar. Examples: {Burak, Dustin, Gary, Jeff, Jeremy, Joseph}, {Burak $\times 4$, Dustin $\times 2$}
C) $\left(\!\binom{6}{18}\!\right)$: Every student votes for a candy bar. We care only about how many votes each type gets. Example: {Butterfinger $\times 15$, Hershey Bar $\times 3$}.
D) $\left(\!\binom{6}{12}\!\right)$: Same as $\left(\!\binom{6}{18}\!\right)$, but each candy bar must get at least one vote. Alternatively, the 6 candy bars are shared by 18 students. How many students end up sharing each of the candy bars? Example: {3 Musketeers, KitKat, Milky Way, Snickers, Butterfinger $\times 11$, Hershey Bar $\times 3$}.
E) $(18)_6$: Six different candy bars are given out, to distinct students. We care about who got which candy bar. Example: {Burak gets the 3 Musketeers, Dustin gets the Butterfinger, Gary gets the Hershey Bar, Jeff gets the KitKat, Jeremy gets the Milky Way, Joseph gets the Snickers}.
F) 18^6: Same as $(18)_6$, but a student may get more than one candy bar. Example: {Burak gets the Butterfinger, Dustin gets the other five candy bars}.
G) 6^{18}: Each student picks their favorite candy bar, and we care about who picked what. Example: {Burak picked Butterfinger, the other 17 students picked Snickers}.
H) $6!S(18,6)$: Same as $\left(\!\binom{6}{12}\!\right)$, except now we keep track of individual students. The six bars are shared by the 18 students. Which students ended up sharing each of the bars? Example: {Burak gets 3 Musketeers, Dustin gets Butterfinger, Gary gets Hershey Bar, Jeff gets KitKat, Jeremy gets Milky Way, the other 13 students shared the Snickers}.
I) $S(18,6)$: Same as $6!S(18,6)$, but we only care about the groups, not which candy each group got. Example: {Burak, Dustin, Gary, Jeff, and Jeremy each got their own bar; the remaining 13 people shared a single bar}.
J) $p_6(18)$: Same as $S(18,6)$, but now we only care about the size of the groups, not who is in them. Example: {five lucky students got their own candy bar; the remaining thirteen had to share a single bar}.
K) $S(18,1)+S(18,2)+S(18,3)+S(18,4)+S(18,5)+S(18,6)$: Same as $S(18,6)$, except now some candy bars can get thrown out uneaten. Example: {Burak got his own bar; the remaining 17 people shared a bar}.

Exam statistics: Low grade=27(F); Median grade=36(C); High grade=48(A)