Math 579 Exam 5 Solutions

1. First, find the number of compositions of 100 into even parts. Then, find the number of integer partitions of 100 into even parts. Finally, find the number of integer partitions of 100 into three distinct even parts.

   First, we divide everything in half so we are taking compositions/partitions of 50 into integer parts. The answers are therefore 2^{49} and $p(50)$ for the first two questions, respectively. For the last question, we have $a_1 + a_2 + a_3 = 50, a_1 > a_2 > a_3$. We set $b_3 = a_3, b_2 = a_2 - 1, b_1 = a_1 - 2$, and have an isomorphism between what we want, and a standard partition $b_1 \geq b_2 \geq b_3$. $b_1 + b_2 + b_3 = a_1 + a_2 + a_3 - 3 = 47$. Hence, the answer to the last question is $p_3(47)$.

2. (10 points) For all $n \geq 1$, prove that $B(n) \geq n$.

   We prove this by induction on n; for $n = 1$, we have $B(1) = 1$. Theorem (5.3) in the text, together with the induction hypothesis, gives us $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i) \geq \binom{n}{1} B(1) + \binom{n}{n} B(n) \geq B(1) + B(n) \geq 1 + n$.

3. (10 points) For all $n \geq 2$, prove that $S(3n, n) > (n!)^2$.

   We will write down each number in $[3n]$ in a $3 \times n$ matrix as follows. In the first row, write $1, 2, \ldots n$ in order. In the second row, write $n+1, n+2, \ldots, 2n$ in any of the $n!$ possible orders. In the third row, write $2n+1, 2n+2, \ldots, 3n$ in any of the $n!$ possible orders. The n columns of this matrix yield a partition of $[3n]$ into n blocks. All $(n!)(n!)$ possible matrices yield different set partitions, so $S(3n, n) \geq (n!)^2$. To prove the strict inequality, we need at least one set partition that wasn't counted. Put each of $1, 2, \ldots, n-1$ into its own block, and put all of $n, n+1, \ldots, 3n$ into one big block. This is a set partition of $[3n]$ into n blocks that was not counted in the matrices above.

4. (10 points) Find a formula for $S(n, 2)$.

   We begin by considering all functions from $[n]$ to two distinct boxes. There are 2^n of them, of which two are not surjective. Hence, there are $2^n - 2$ surjective functions onto two distinct boxes. We now call two of these functions equivalent if one is obtained from the other by relabeling the two boxes. All equivalence classes have size two, hence the number of surjective functions onto two indistinct boxes is $(2^n - 2)/2 = 2^{n-1} - 1$. This works for $n \geq 1$; if $n = 0$, however, $S(0, 2) = 0$.

5. (12 points) Find a formula for $S(n, n-3)$.

   Set partitions of $[n]$ into $n-3$ blocks, come in three varieties:
   (A) One block of size 4, $n-4$ blocks of size 1. There are $\binom{n}{4}$ of these.
   (B) One block of size 3, one block of size 2, $n-5$ blocks of size 1. There are $\binom{n}{3}\binom{n-3}{2}$ of these.

(C) Three blocks of size 2, $n-6$ blocks of size 1. If we consider the three bigger blocks different, there are $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}$ of these. However, these three are actually indistinguishable, so there are actually $\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}/3!$ of these.

Hence, putting it all together, $S(n,3)=\binom{n}{4}+\binom{n}{3}\binom{n-3}{2}+\frac{1}{3!}\binom{n}{2}\binom{n-2}{2}\binom{n-4}{2}$.

Part II. Consider all partitions of $n : (a_1, a_2, a_3, \ldots)$, such that $a_1 \geq a_2 \geq a_3 \geq \cdots \geq 1$. Let $f(n)$ denote the number of such partitions that are self-conjugate and have $a_3 = 3$. Find a closed form for $f(n)$; partition functions p, p_k are considered closed form.

SOLUTION 1: Draw the Ferrers diagram for the partition. Decompose it into (symmetric) hooks. Remove the first row and first column (the first hook, of odd length). Remove the first row and first column from what's left (the second hook, of odd length). At this point a single block will remain, because $a_3 = 3$.

Hence, in order to have any self-conjugate partitions with $a_3 = 3$, we need to be able to write $n = (2x+1)+(2y+1)+1$, where $2x+1$ is the odd length of the first hook, and $2y+1$ is the odd length of the second hook. Further, these must satisfy $x > y \geq 1$, since the three hooks must be of different lengths. We rewrite $n-3 = 2(x+y)$, and again $(n-3)/2 = x+y$. Hence, $f(n) =$ the number of partitions of $(n-3)/2$ into two parts of different sizes.

To fix this last point, set $z = x-1$. Now, $z \geq y \geq 1$, so we are counting ordinary partitions into two parts of $z+y = (x-1)+y = (n-5)/2$. Hence, $f(n) = p_2(\frac{n-5}{2})$. This is valid whenever $\frac{n-5}{2}$ is an integer (that is, for n odd). For n even, there are NO such partitions.

SOLUTION 2: The Ferrers diagram must have a 3×3 grid in the northwest corner; the remaining $n-9$ blocks are divided in half by the symmetry imposed by the self-conjugate condition (in particular, if n is even, $f(n) = 0$). These $(n-9)/2$ blocks in the first two rows are divided between the first and second row, with the restriction that the first row must have no fewer blocks than the second. $p_2(\frac{n-9}{2})$ is almost right, except this does not allow all $(n-9)/2$ blocks to be in the first row. Hence, $f(n) = p_2(\frac{n-9}{2})+1$ for odd $n \geq 9$ and $f(n) = 0$ otherwise.

$f(1) = f(2) = f(3) = f(4) = f(5) = f(6) = f(7) = f(8) = f(10) = f(12) = f(14) = 0$
$f(9) = f(11) = 1 \quad f(13) = f(15) = 2$

Exam statistics: Low grade=28(F); Median grade=35.5(C/C-); High grade=49(A)