Math 579 Exam 4 (part I): 3/1/7

Please read the exam instructions.

Please write your answers on **separate paper**, indicate clearly what work goes with which problem, and put your name on every sheet. Cross out work you do not wish graded; incorrect work can lower your grade, even compared with no work at all. Keep this list of problems for your records. Show all necessary work in your solutions; if you are unsure, show it. Each problem is worth a minimum of 5 points, and a maximum that is indicated. You have 40 minutes. *Choose three problems.*

1. (8 points) How many subsets of $[n]$ are larger than their complements?

2. (10 points) Evaluate the sum $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$.

3. (10 points) Let $n \in \mathbb{N}$. Prove that $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$.

4. (10 points) We may write $x^4 = (x)_4 + 6(x)_3 + a(x)_2 + (x)_1$, for some integer constant a. First, find a. Then, use the difference calculus to evaluate in closed form $\sum_{k=0}^{n} k^4$.

5. (12 points) Let p be prime. Prove that p divides $\binom{p-1}{k} + (-1)^{k+1}$, for all k satisfying $0 \leq k \leq p-1$.
   HINT: Start by proving that p divides $\binom{p}{k}$ for all k with $1 \leq k \leq p-1$.