1. How many subsets of \([n]\) are larger than their complements?

We pair off the subsets of \([n]\) by pairing each subset with its complement. If \(n\) is odd, then one of these will always be larger than the other, so exactly half of the subsets \((2^n/2 = 2^{n-1})\) are larger than their complements. If \(n\) is even, this same strategy works, EXCEPT for those subsets of size \(n/2\). There are \(\binom{n}{n/2}\) of them, which are divided into half as many pairs. Each of these pairs needs to be subtracted off; hence the answer for even \(n\) is \(2^{n-1} - \frac{1}{2}\binom{n}{n/2}\). The book has a typo in its solution.

2. Evaluate the sum \(\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k}\).

Solution 1: We use the identity \(\binom{n}{k} = \binom{n-1}{k-1}\), which is valid for \(k \neq 0\). In our case, this condition holds, so we may rewrite the sum to be \(\sum_{k=0}^{n} \frac{1}{k+1} \binom{n+1}{k+1} = \frac{1}{n+1} \sum_{k=0}^{n} \binom{n+1}{k+1} = \frac{1}{n+1} \sum_{j=1}^{n+1} \binom{n+1}{j} = \frac{1}{n+1} \left(-1 + \sum_{j=0}^{n+1} \binom{n+1}{j}\right) = \frac{1}{n+1} (-1 + 2^{n+1})\).

Solution 2: We begin with \((1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k\). We integrate both sides, from 0 to \(y\), to get \(\frac{1}{n+1}(1 + x)^{n+1}|_0^y = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} x^{k+1}|_0^y\). We simplify to get \(\frac{1}{n+1}((1 + y)^{n+1} - 1) = \sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} y^{k+1}\). Plug in \(y = 1\), and we find \(\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{1}{n+1}(2^{n+1} - 1)\).

3. Let \(n \in \mathbb{N}\). Prove that \(\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2\).

Solution 1: Because \(n \in \mathbb{N}\), we may rewrite \(\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}\), and give a combinatorial proof. Consider \(2n\) students, half male and half female. Let \(S\) be the number of ways to select \(n\) of these students. \(|S| = \binom{2n}{n}\), by ignoring gender. But we may also select \(n\) students by first deciding to choose \(k\) male students and hence \(n-k\) female students. There are \(\binom{n}{k} \binom{n}{n-k}\) to choose these students. We could do this for any \(k\) with \(0 \leq k \leq n\), hence \(|S| = \sum_{k=0}^{n} \binom{n}{k} \binom{n}{n-k}\).

Solution 2: This uses Thm 4.7 in the text, which states that: \(\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i}\). This requires \(n, m, k\) to all be positive integers. We fix \(n \in \mathbb{N}\), and set \(k = n, m = n\). The theorem then simplifies to \(\binom{2n}{n} = \sum_{i=0}^{n} \binom{n}{i} \binom{n}{n-i}\). This solves the problem, since again \(\binom{n}{n-i} = \binom{n}{i}\).
4. We may write \( x^4 = (x)_4 + 6(x)_3 + a(x)_2 + (x)_1 \), for some integer constant \( a \). First, find \( a \). Then, use the difference calculus to evaluate in closed form \( \sum_{k=0}^{n} k^4 \).

A cute way to find \( a \) is to plug in a value of \( x \). For example, \( x = 2 \) gives \( 2^4 = (2)_4 + 6(2)_3 + a(2)_2 + (2)_1 = 0 + 0 + a - 2 + 2; \) hence \( a = 7 \). Alternatively, we can just expand: \( (x)_4 + 6(x)_3 + a(x)_2 + (x)_1 = x(x-1)(x-2)(x-3) + 6x(x-1)(x-2) + ax(x-1) + x = x^4 + (a - 7)x^2 + (7 - a)x \). Hence \( a = 7 \).

Now, \( \sum_{0 \leq k < n+1} k^4 = \sum_{0 \leq k < n+1} (k)_4 + 6(k)_3 + 7(k)_2 + (k)_1 \). We apply the fundamental theorem of difference calculus to get \( \frac{1}{5}(k)_5 + \frac{6}{4}(k)_4 + \frac{7}{3}(k)_3 + \frac{1}{2}(k)_2 |_{k=0}^{k=n+1} = \frac{1}{5}(n+1)_5 + \frac{6}{4}(n+1)_4 + \frac{7}{3}(n+1)_3 + \frac{1}{2}(n+1)_2 - \left( \frac{1}{5}(0)_5 + \frac{6}{4}(0)_4 + \frac{7}{3}(0)_3 + \frac{1}{2}(0)_2 \right) \). If you want to simplify this, it’s \( \frac{6n^5 + 15n^4 + 10n^3 - n}{30} \).

5. Let \( p \) be prime. Prove that \( p \) divides \( \binom{p-1}{k} + (-1)^{k+1} \), for all \( k \) satisfying \( 0 \leq k \leq p - 1 \).

HINT: Start by proving that \( p \) divides \( \binom{p}{k} \) for all \( k \) with \( 1 \leq k \leq p - 1 \).

\( \binom{p}{k} = \frac{p!}{k!(p-k)!} \), since \( p > k > 0 \). The denominator is the product of many factors, all strictly less than \( p \); hence \( p \) does not divide the denominator. Since prime \( p \) divides the numerator, \( p \) must divide \( \binom{p}{k} \). Now, \( \binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1} \). Hence, consecutive terms in the \( (p - 1) \)th row of Pascal’s triangle sum to a multiple of \( p \). We finish the proof by induction on \( k \). For \( k = 0 \), \( \binom{p-1}{0} = 1 \) so \( p \) divides \( \binom{p-1}{0} - 1 \). For \( k > 0 \), \( p \) divides \( \binom{p}{k} = \binom{p-1}{k} + (-1)^k \), and \( p \) also divides \( \binom{p}{k-1} + (-1)^k \) by the inductive hypothesis. Hence \( p \) must also divide \( (-1)^k + \binom{p-1}{k} \).

Part II. Guess a closed form for \( f(n) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{(n-1)^{n}} \), for all integer \( n \geq 2 \). Then, prove your formula.

We first calculate \( f(2) = \frac{1}{2}, f(3) = \frac{3}{2}, f(4) = \frac{3}{4} \). This inspires the guess that \( f(n) = \frac{n-1}{n} \). This is a common technique in mathematics, to try small cases to try to find a pattern. Calculating \( f(n) \) for \( n = 2, 3, 4 \), and guessing \( f(n) = \frac{n-1}{n} \), would be worth much of the credit for the problem. Of course, almost nobody got that far and stopped; once you have the guess it is straightforward to prove it by induction.

The base case is already complete. We start with \( \frac{n-1}{n} = f(n) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^3} + \cdots + \frac{1}{(n-1)^{n}} \). We add \( \frac{1}{n(n+1)} \) to both sides; the right hand side is \( f(n+1) \) by definition. The left hand side is \( \frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1) + 1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1} \). Hence \( f(n+1) = \frac{n}{n+1} \), as desired.

Exam statistics: Low grade=27(F); Median grade=33(D) (ouch!); High grade=47(A)