# Math 579 Exam 4 Solutions

1. How many subsets of $[n]$ are larger than their complements?

   We pair off the subsets of $[n]$ by pairing each subset with its complement. If $n$ is odd, then one of these will always be larger than the other, so exactly half of the subsets ($2^n/2 = 2^{n-1}$) are larger than their complements. If $n$ is even, this same strategy works, EXCEPT for those subsets of size $n/2$. There are $\binom{n}{n/2}$ of them, which are divided into half as many pairs. Each of these pairs needs to be subtracted off; hence the answer for even $n$ is $2^{n-1} - \frac{1}{2}\binom{n}{n/2}$. The book has a typo in its solution.

2. Evaluate the sum $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}$.

   Solution 1: We use the identity $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$, which is valid for $k \neq 0$. In our case, this condition holds, so we may rewrite the sum to be $\sum_{k=0}^{n} \frac{1}{n+1}\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{k=0}^{n}\binom{n+1}{k+1} = \frac{1}{n+1}\sum_{j=1}^{n+1}\binom{n+1}{j} = \frac{1}{n+1}\left(-1 + \sum_{j=0}^{n+1}\binom{n+1}{j}\right) = \frac{1}{n+1}\left(-1 + 2^{n+1}\right)$

   Solution 2: We begin with $(1+x)^n = \sum_k \binom{n}{k}x^k$. We integrate both sides, from 0 to $y$, to get $\frac{1}{n+1}(1+x)^{n+1}\big|_0^y = \sum_k \frac{1}{k+1}\binom{n}{k}x^{k+1}\big|_0^y$. We simplify to get $\frac{1}{n+1}\left((1+y)^{n+1} - 1\right) = \sum_k \frac{1}{k+1}\binom{n}{k}y^{k+1}$. Plug in $y = 1$, and we find $\sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k} = \frac{1}{n+1}(2^{n+1} - 1)$.

3. Let $n \in \mathbb{N}$. Prove that $\binom{2n}{n} = \sum_{k=0}^{n}\binom{n}{k}^2$.

   Solution 1: Because $n \in \mathbb{N}$, we may rewrite $\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$, and give a combinatorial proof. Consider $2n$ students, half male and half female. Let $S$ be the number of ways to select $n$ of these students. $|S| = \binom{2n}{n}$, by ignoring gender. But we may also select $n$ students by first deciding to choose $k$ male students and hence $n-k$ female students. There are $\binom{n}{k}\binom{n}{n-k}$ to choose these students. We could do this for any $k$ with $0 \leq k \leq n$, hence $|S| = \sum_{k=0}^{n}\binom{n}{k}\binom{n}{n-k}$.

   Solution 2: This uses Thm 4.7 in the text, which states that: $\binom{n+m}{k} = \sum_{i=0}^{k}\binom{n}{i}\binom{m}{k-i}$. This requires $n, m, k$ to all be positive integers. We fix $n \in \mathbb{N}$, and set $k = n, m = n$. The theorem then simplifies to $\binom{2n}{n} = \sum_{i=0}^{n}\binom{n}{i}\binom{n}{n-i}$. This solves the problem, since again $\binom{n}{n-i} = \binom{n}{i}$.

4. We may write $x^4 = (x)_4 + 6(x)_3 + a(x)_2 + (x)_1$, for some integer constant $a$. First, find $a$. Then, use the difference calculus to evaluate in closed form $\sum_{k=0}^{n} k^4$.

   A cute way to find $a$ is to plug in a value of $x$. For example, $x = 2$ gives $2^4 = (2)_4+6(2)_3+a(2)_2+(2)_1 = 0+0+a\cdot 2+2$; hence $a = 7$. Alternatively, we can just expand: $(x)_4+6(x)_3+a(x)_2+(x)_1 = x(x-1)(x-2)(x-3)+6x(x-1)(x-2)+ax(x-1)+x = x^4 + (a-7)x^2 + (7-a)x$. Hence $a = 7$.

   Now, $\sum_{0\le k<n+1} k^4 = \sum_{0\le k<n+1} (k)_4 + 6(k)_3 + 7(k)_2 + (k)_1$. We apply the fundamental theorem of difference calculus to get $\frac{1}{5}(k)_5 + \frac{6}{4}(k)_4 + \frac{7}{3}(k)_3 + \frac{1}{2}(k)_2|_{k=0}^{k=n+1} = \frac{1}{5}(n+1)_5 + \frac{6}{4}(n+1)_4 + \frac{7}{3}(n+1)_3 + \frac{1}{2}(n+1)_2 - (\frac{1}{5}(0)_5 + \frac{6}{4}(0)_4 + \frac{7}{3}(0)_3 + \frac{1}{2}(0)_2)$. If you want to simplify this, it's $\frac{6n^5+15n^4+10n^3-n}{30}$.

5. Let $p$ be prime. Prove that $p$ divides $\binom{p-1}{k} + (-1)^{k+1}$, for all $k$ satisfying $0 \le k \le p-1$. HINT: Start by proving that $p$ divides $\binom{p}{k}$ for all $k$ with $1 \le k \le p-1$.

   $\binom{p}{k} = \frac{p!}{k!(p-k)!}$, since $p > k > 0$. The denominator is the product of many factors, all strictly less than $p$; hence $p$ does not divide the denominator. Since prime $p$ divides the numerator, $p$ must divide $\binom{p}{k}$. Now, $\binom{p}{k} = \binom{p-1}{k} + \binom{p-1}{k-1}$. Hence, consecutive terms in the $(p-1)^{\text{th}}$ row of Pascal's triangle sum to a multiple of $p$. We finish the proof by induction on $k$. For $k = 0$, $\binom{p-1}{0} = 1$ so $p$ divides $\binom{p-1}{0} - 1$. For $k > 0$, $p$ divides $\binom{p}{k} = \binom{p-1}{k-1} + (-1)^k - (-1)^k + \binom{p-1}{k}$, and $p$ also divides $\binom{p-1}{k-1} + (-1)^k$ by the inductive hypothesis. Hence $p$ must also divide $-(-1)^k + \binom{p-1}{k} = \binom{p-1}{k} + (-1)^{k+1}$.

Part II. Guess a closed form for $f(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n}$, for all integer $n \ge 2$. Then, prove your formula.

We first calculate $f(2) = \frac{1}{2}, f(3) = \frac{2}{3}, f(4) = \frac{3}{4}$. This inspires the guess that $f(n) = \frac{n-1}{n}$. This is a common technique in mathematics, to try small cases to try to find a pattern. Calculating $f(n)$ for $n = 2, 3, 4$, and guessing $f(n) = \frac{n-1}{n}$, would be worth much of the credit for the problem. Of course, almost nobody got that far and stopped; once you have the guess it is straightforward to prove it by induction.

The base case is already complete. We start with $\frac{n-1}{n} = f(n) = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{(n-1)\cdot n}$. We add $\frac{1}{n\cdot(n+1)}$ to both sides; the right hand side is $f(n+1)$ by definition. The left hand side is $\frac{n-1}{n} + \frac{1}{n(n+1)} = \frac{(n-1)(n+1)}{n(n+1)} + \frac{1}{n(n+1)} = \frac{n^2}{n(n+1)} = \frac{n}{n+1}$. Hence $f(n+1) = \frac{n}{n+1}$, as desired.

Exam statistics: Low grade=27(F); Median grade=33(D) (ouch!); High grade=47(A)