# Math 579 Exam 2 Solutions

1. Let $a_0 = 1$, and let $a_{n+1} = 3a_n + 6$ for all nonnegative integers $n$. Prove that $a_n = 4 \cdot 3^n - 3$.

   Base case: $1 = 4 - 3 = 4 \cdot 3^0 - 3$; hence theorem holds for $n = 0$.
   Assume theorem holds up to some $n \geq 0$. By definition, $a_{n+1} = 3a_n + 6$. By the inductive hypothesis, $a_n = 4 \cdot 3^n - 3$, so $a_{n+1} = 3(4 \cdot 3^n - 3) + 6 = 4 \cdot 3^{n+1} - 9 + 6 = 4 \cdot 3^{n+1} - 3$, as desired.

2. Prove that for all positive integers $n$, we have $1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$.

   Base case: $1^3 = 1 = (\frac{1(2)}{2})^2$; hence theorem holds for $n = 1$.
   Assume theorem holds up to some $n \geq 1$. Hence $1^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2}\right)^2$. We add $(n+1)^3$ to both sides, to get $1^3 + \cdots + n^3 + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$. The right hand side is $(1/4)(n^2(n+1)^2 + 4(n+1)^3) = (1/4)(n+1)^2(n^2 + 4(n+1)) = (1/4)(n+1)^2(n+2)^2$, as desired.

3. Prove that a positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9.

   Write the number as $X = a_n 10^{n-1} + a_{n-1}10^{n-2} + \cdots + a_2 10^1 + a_1$, and the sum of its digits as $Y = a_n + a_{n-1} + \cdots + a_1$. We will prove the stronger statement that $9|(X - Y)$. Proof by induction on $n$, the number of digits. If $n = 1$, then $X = Y$, so the theorem holds. Assume the theorem holds up to $n - 1$. Set $X' = (X - a_1)/10$, and $Y' = (Y - a_1)$. $X'$ is an $(n-1)$-digit number, the sum of whose digits is $Y'$. By the inductive hypothesis, $9|(X' - Y')$. Certainly $9|9X'$, so $9|(X' - Y' + 9X')$ and hence $9|(10X' - Y')$. But we see that $10X' - Y' = X - Y$, as desired.

   For the next two problems, recall that the Fibonacci numbers are defined as $F_1 = F_2 = 1, F_{n+2} = F_{n+1} + F_n$ for nonnegative integer $n$.

4. Prove that $F_m \geq (1.3)^m$, for all integer $m \geq 4$.

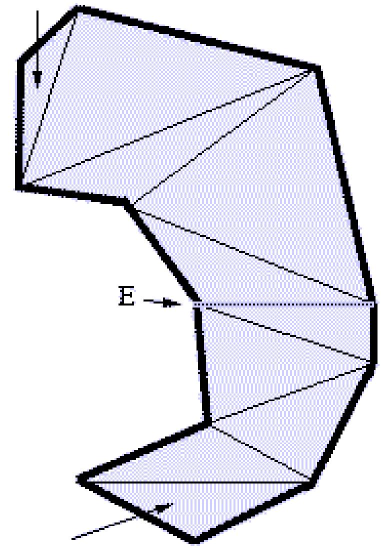
   $F_4 = 3 \geq 2.8561 = 1.3^4$. Also, $F_4 = 5 \geq 3.71293 = 1.3^5$. Now, assume the theorem holds up to $m+1$. $F_{m+2} = F_{m+1} + F_m$, by definition. By the strong inductive hypothesis, twice, we have $F_{m+1} \geq 1.3^{m+1} + 1.3^m = 1.3^m(1.3 + 1) = 1.3^m(2.3) > 1.3^m(1.69) = 1.3^m 1.3^2 = 1.3^{m+2}$, as desired.

5. Prove that $F_1^2 + F_2^2 + \cdots + F_m^2 = F_m F_{m+1}$ for all natural $m$.

   For $m = 1$, we have $F_1^2 = 1^2 = 1$, while $F_1 F_2 = 1^2 = 1$, so the theorem holds.
   Now, assume the theorem holds up to $m$. Hence $F_1^2 + F_2^2 + \cdots + F_m^2 = F_m F_{m+1}$. We add $F_{m+1}^2$ to both sides, to get $F_1^2 + F_2^2 + \cdots + F_m^2 + F_{m+1}^2 = F_m F_{m+1} + F_{m+1}^2 = F_{m+1}(F_m + F_{m+1}) = F_{m+1}F_{m+2}$, as desired.

Part II: A *simple* polygon has an inside and an outside. To *triangulate* a simple polygon is to add noncrossing interior edges between existing vertices, so that the inside of the polygon is partitioned into triangles. An *exterior* triangle is one that has two sides on the exterior of the polygon.

Prove that, for any simple polygon with at least four sides, and any triangulation, that at least two of the created triangles will be exterior.




Proof by strong induction on the number of edges. Base case: four edges; there are two ways to triangulate such a polygon, and both triangles formed are exterior.
General case: Let $P$ be any simple polygon with more than four edges, and we fix any triangulation of it. Choose any of these new edges $E$; this divides $P$ into two simple polygons (say $P_1$ and $P_2$), each of which has fewer sides than $P$ (but at least three sides). The triangulation of $P$ is inherited by $P_1$ and $P_2$; that is, these edges are between vertices, noncrossing, and partition the interiors of $P_1, P_2$ into triangles. If either of $P_1, P_2$ has exactly three sides, then it is itself an exterior triangle of $P$. Otherwise, by the inductive hypothesis, each of $P_1$ and $P_2$ has two exterior triangles in its triangulation (which is the same as the triangulation of $P$). Although the triangulations are the same in the big polygon as in the smaller ones, triangles that are exterior in the smaller polygons might not be exterior in the big polygon. It is possible that one exterior triangle of $P_1$ has a side on $E$, which is an exterior side of $P_1$, but is interior to $P$. Similarly, one of the exterior triangles of $P_2$ might be lost this way. In the diagram, this is precisely what happens. However, each of $P_1$ and $P_2$ must contribute at least one exterior triangle that is not lost (if either has exactly three sides, then it equally contributes one exterior triangle), hence $P$ has two exterior triangles, one each from $P_1$ and $P_2$, as desired.

Exam statistics: Low grade=33(D); Median grade=40.5(B/B-); High grade=52(A+)
(of course, these are subject to change due to extra credit)