1. Prove that \( b_n = 3^n \) satisfies the recurrence relation \( a_n = 2a_{n-1} + 3a_{n-2} \).

We compute \( 2b_{n-1} + 3b_{n-2} = 2 \cdot 3^{n-1} + 3 \cdot 3^{n-2} = 3^{n-2}(2 \cdot 3 + 3) = 3^{n-2}(9) = 3^n = b_n \).

2. Consider the recurrence given by \( a_0 = 0, a_1 = 0, a_2 = 12, a_n = -3a_{n-1} + 4a_{n-3} + 18 \) \((n \geq 3)\). Solve this using the methods of our packet.

We start by considering the homogeneous recurrence relation \( a_n = -3a_{n-1} + 4a_{n-3} \), with characteristic polynomial \( x^3 + 3x^2 - 4 = (x - 1)(x + 2)^2 \).

Hence the general solution to the homogeneous problem is \( a_n = A + B(-2)^n + Cn(-2)^n \).

We turn now to the nonhomogeneous problem. The nonhomogeneous term is a polynomial of degree 0, but all such are already included in the nonhomogeneous case. Hence we instead guess \( a_n = Dn \) as a solution. We get \( Dn = -3D(n - 1) + 4D(n - 3) + 18 \).

The \( n \) terms all cancel, leaving \( 0 = 3D - 12D + 18 \), so \( D = 2 \).

The general nonhomogeneous solution is \( a_n = 2n + A + B(-2)^n + Cn(-2)^n \).

We now use our initial conditions: \( 0 = a_0 = 2 \cdot 0 + A + B(-2)^0 + C \cdot 0(-2)^0, 0 = a_1 = 2 \cdot 1 + A + B(-2)^1 + C \cdot 1(-2)^1, 12 = a_2 = 2 \cdot 2 + A + B(-2)^2 + C \cdot 2(-2)^2 \).

This gives system of equations \( \{0 = A + B, 0 = 2 + A - 2B - 2C, 12 = 4 + A + 4B + 8C\} \).

This has solution \( A = 0, B = 0, C = 1 \).

Hence, the solution we seek is \( a_n = 2n + n(-2)^n = n(2 + (-2)^n) \).

3. Consider the recurrence given by \( a_0 = 0, a_1 = 0, a_2 = 12, a_n = -3a_{n-1} + 4a_{n-3} + 18 \) \((n \geq 3)\). Solve this using generating functions.

We set \( A(x) = \sum_{n \geq 0} a_n x^n \). Multiplying our relation by \( x^n \) and summing over \( n \geq 3 \), we get \( \sum_{n \geq 3} a_n x^n = -3 \sum_{n \geq 3} a_{n-1} x^n + 4 \sum_{n \geq 3} a_{n-3} x^n + 18 \sum_{n \geq 3} x^n \). Hence \( A(x) - 0 - 0x - 12x^2 = -3x(A(x) - 0 - 0x) + 4x^2A(x) + 18x^3 \frac{1}{1-x} \).

We rearrange as \( A(x)(1 + 3x - 4x^3) = 12x^2 + \frac{18x^3}{1-x} \).

Hence \( A(x) = \frac{12x^2 + 6x^3}{(1-x)(1+3x-4x^3)} = \frac{12x^2 + 6x^3}{(1-x)^2(1+2x)^2} \).

We now have a partial fractions problem of \( A(x) = \frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{1+2x} + \frac{D}{(1+2x)^2} \), or \( 6x^2(2+x) = A(1-x)(1+2x)^2 + B(1+2x)^2 + C(1-x)^2(1+2x) + D(1-x)^2 \).

Taking \( x = 1 \), we get \( 18 = B(1+2)^2 \), or \( B = 2 \). Taking \( x = -\frac{1}{2} \), we get \( 6(\frac{3}{4})\frac{1}{2} = D(1-\frac{1}{2})^2 \), or \( D = 1 \). Taking \( x = 0 \), we get \( 0 = A + B + C + D \), or \( 0 = A + C + 3 \). Taking \( x = -1 \), we get \( 6(1)(1) = A(2)(-1)^2 + B(-1)^2 + C(2)^2(-1) + D(2)^2 = 2A + B - 4C + 4D = 2A - 4C + 6 \), or \( 0 = A - 2C \). Solving, we get \( A = -2, C = -1 \).

Hence, \( A(x) = -2 \sum_{n \geq 0} x^n + 2 \sum_{n \geq 0} (n+1)x^n - \sum_{n \geq 0} (-2)^n x^n + \sum_{n \geq 0} (n+1)(-2)^n x^n = \sum_{n \geq 0} (-2 + 2(n+1) - (-2)^n + (n+1)(-2)^n) x^n \).

Hence \( a_n = -2 + 2(n+1) - (-2)^n + (n+1)(-2)^n = 2n + n(-2)^n = n(2 + (-2)^n) \).