1. Use methods of difference calculus to compute $\sum_{i=1}^{25} i^4$.

We first translate $\sum_{i=1}^{25} i^4 = \sum_{i=1}^{26} x^4 \delta x$. Now, we compute Stirling numbers of the second kind to find $x^4 = S(4, 4)x^4 + S(4, 3)x^3 + S(4, 2)x^2 + S(4, 1)x^1 = x^4 + 6x^3 + 7x^2 + x^1$. Hence our sum is $\sum_{i=1}^{26} x^4 + 6x^3 + 7x^2 + x^1 = \frac{1}{2}x^6 + \frac{9}{2}x^5 + \frac{7}{2}x^4 + \frac{1}{2}x^2(11) = \frac{1}{2}26^3 + \frac{9}{2}26^2 + \frac{7}{2}26 + \frac{1}{2}(11^2 + \frac{9}{2}11 + \frac{7}{2}1) = 2, 153, 645$.

2. Let $u, v$ be functions from $\mathbb{Z}$ to $\mathbb{R}$. Prove that $\Delta(uv) = u\Delta v + Ev\Delta u$.

We calculate $E\nu \Delta u + u \Delta v = (v(x+1)\Delta u + u(x)\Delta v = v(x+1)(u(x+1) - u(x)) + u(x)(v(x+1) - v(x)) = u(x+1)v(x+1) - u(x)v(x+1) + u(x)v(x+1) - u(x)v(x)$.

Two terms cancel, leaving $u(x+1)v(x+1) - u(x)v(x)$.

3. Let $n \in \mathbb{N}_0$. Calculate and simplify $\sum_{i=0}^{n} x^1 x^\frac{i}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15}$.

Warning: $x^\frac{1}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15}$.

Method 1: Write $x^i = x = (x - 10 + 10)$. Hence, $\sum_{i=0}^{n} x^I x^\frac{i}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15} = \sum_{i=0}^{n} (x - 10 + 10) x^\frac{i}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15}$.

Method 2: Summation by parts. Set $u = x, \Delta u = x^\frac{1}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15}$. We have $\Delta u = x^\frac{1}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15}$. Now, $\sum_{i=0}^{n} x^\frac{i}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15} = \sum_{i=0}^{n} x^\frac{i}{12} x^\frac{1}{13} x^\frac{1}{14} x^\frac{1}{15}$.

Let $f, g$ be functions from $\mathbb{Z}$ to $\mathbb{R}$. Suppose that $\Delta f = \Delta g$. Prove that there is some constant $C$ such that $f(x) = g(x) + C$.

Lemma: If $\Delta h = 0$, then there is some constant $C$ with $h(x) = C$.

Proof: Set $C = h(0)$. We prove $\forall n \in \mathbb{N}_0$, $h(n) = C$ by induction. Base case: $h(0) = C$ already. Now, assume that $h(n) = C$. We have $0 = \Delta h = h(n+1) - h(n)$, so $h(n+1) = h(n) = C$. A similar proof works for all negative integer $n$.

Now, set $h(x) = f(x) - g(x)$. We have $\Delta h = f(x+1) - g(x+1) - (f(x) - g(x)) = (f(x+1) - f(x)) - (g(x+1) - g(x)) = \Delta f - \Delta g = 0 - 0 = 0$. Hence, by lemma, there is some constant $C$ with $h(x) = C$. So, $f(x) - g(x) = C$, which rearranges to $f(x) = g(x) + C$.

5. Let $n \in \mathbb{N}$. Calculate $\sum_{i=1}^{n} H_i x^\delta x$.

We rewrite $\sum_{i=1}^{n} H_i x^\delta x = \sum_{i=1}^{n} x^H H_i x^\delta x$, and use summation by parts. Set $u = H_i$, $\Delta v = x^H$. We have $\Delta u = x^\frac{1}{10}$ and $v = x^\frac{1}{10}$. Hence, $\sum_{i=1}^{n} x^H H_i x^\delta x = x^H H_1 x^\delta x - \sum_{i=1}^{n} (x+1)^\frac{1}{10} x^\frac{i}{12} x^\frac{i}{13} x^\frac{i}{14} x^\frac{i}{15} (nH_n - 1H_1) - \sum_{i=1}^{n} x^H H_i x^\delta x = nH_n - 1H_1 - nH_n - (n-1) = nH_n - n$.

6. Recall that $x^m = \left\{ \begin{array}{ll} x(x+1) \cdots (x+m-1) & m \geq 0 \\ \frac{1}{(x-1)(x-2) \cdots (x+m)} & m \leq 0 \end{array} \right.$ Define the “other” difference operator $\Delta'$ as $\Delta' f = f(x) - f(x-1)$. Compute and simplify $\Delta' x^m$.

For $m \geq 1$, we have $\Delta' x^m = x(x+1) \cdots (x+m-2)(x+m-1) - (x-1)(x) \cdots (x+m-2) = x(x+1) \cdots (x+m-2)[x+m-1 - (x-1)] = mx^{m-1}$. 

For $m \leq 0$, we have $\Delta' x^m = \frac{1}{(x-1)(x-2) \cdots (x+m)} - \frac{1}{(x-2)(x-3) \cdots (x+m)(x+m-1)} = \frac{x^m - 1}{(x-1)(x-2) \cdots (x+m)(x+m-1)} = mx^{m-1}$.

Hence, in both cases, the result is $\Delta' x^m = mx^{m-1}$. 

MATH 579: Combinatorics

Exam 4 Solutions