MATH 579: Combinatorics

Homework 6 Solutions

1. Find a closed form for the generating function for the sequence $1, -1, 1, -1, 1, -1, \ldots$.
   We have a formula for this: $\sum_{k\geq 0}(-1)^k x^k = \frac{1}{1+x}$.

2. Find a closed form for the generating function for the sequence $0, 0, 0, 1, -1, 1, -1, 1, -1, \ldots$.
   We shift the previous problem by 3, via $x^3\sum_{k\geq 0}(-1)^k x^k = \sum_{k\geq 0}(-1)^k x^{k+3} = \frac{x^3}{1+x}$.

3. Find a closed form for the generating function for the sequence $\sum_{k\geq 0}(7k-2)x^k$.
   We have $\sum_{k\geq 0}(7k-2)x^k = 7\sum_{k\geq 0}kx^k - 2\sum_{k\geq 0}x^k = 7\frac{x}{(1-x)^2} - 2\frac{1}{1-x} = \frac{9x-2}{(1-x)^2}$.

4. Find a closed form for the generating function for the sequence $0, 1, 4, 9, 16, 25, \ldots$.
   Note that $k^2 = 2\binom{k}{2} + k$, so $\sum_{k\geq 0}k^2x^k = \sum_{k\geq 0}2\binom{k}{2} + kx^k = 2\sum_{k\geq 0}\binom{k}{2}x^k + \sum_{k\geq 0}kx^k = 2\frac{x^2}{(1-x)^3} + \frac{x}{(1-x)^2} = \frac{x^2+x}{(1-x)^3}$.

5. Find a closed form for the generating function for the Fibonacci numbers.
   We have $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$ (and $F_0 = 0, F_1 = 1$). We multiply both sides by x^n, and sum over all $n \geq 2$. We get $F(x) - F_0 - F_1x = x(F(x) - F_0) + x^2F(x)$, or $F(x) - x = xF(x) + x^2F(x)$. We solve for $F(x)$, getting $F(x) = \frac{x}{1-x-x^2}$.

6. Solve the recurrence given by $a_0 = 0, a_1 = 1, a_n = 4a_{n-2}$ $(n \geq 2)$ using generating functions.
   We have $a_n = 4a_{n-2}$, for $n \geq 2$. Multiplying both sides by x^n, and sum over all $n \geq 2$. We get $A(x) - a_0 - a_1x = 4x^2A(x)$, or $A(x) - x = 4x^2A(x)$. We solve for $A(x)$, getting $A(x) = \frac{x}{1-4x^2} = \frac{x}{(1-2x)(1+2x)}$. We apply partial fractions, getting $A(x) = \frac{1}{4}\frac{1}{1-2x} - \frac{1}{4}\frac{1}{1+2x} = \frac{1}{4}\sum_{n\geq 0}2^nx^n - \frac{1}{4}\sum_{n\geq 0}(-2)^nx^n$. Hence $a_n = \frac{1}{4}2^n - \frac{1}{4}(-2)^n$.

7. Solve the recurrence given by $a_0 = a_1 = 2, a_n = -2a_{n-1} - a_{n-2}$ $(n \geq 2)$ using generating functions.
   We multiply the relation by x^n and sum over all $n \geq 2$, getting $A(x) - a_0 - a_1x = -2x(A(x) - a_0) - x^2A(x)$, or $A(x) - 2 - 2x = -2xA(x) + 4x - x^2A(x)$. We solve for $A(x)$, getting $A(x) = \frac{6x+2}{1+2x+x^2} = \frac{6x+2}{(1+x)^2}$. Applying partial fractions, we get $A(x) = \frac{6}{1+x} - \frac{4}{(1+x)^2} = 6\sum_{n\geq 0}(-1)^nx^n - 4\sum_{n\geq 0}(n+1)(-1)^nx^n$. Hence $a_n = 6(-1)^n - 4(n+1)(-1)^n = 2(1-2n)(-1)^n$.

8. Solve the recurrence given by $a_0 = a_1 = 0, a_n = a_{n-1} + 2a_{n-2} + 3$ $(n \geq 2)$ using generating functions.
   We multiply the relation by x^n and sum over all $n \geq 2$, getting $A(x) - a_0 - a_1x = x(A(x) - a_0) + 2x^2A(x) + 3\sum_{n\geq 2}x^n$, or $A(x) = xA(x) + 2x^2A(x) + 3(\frac{1}{1-x} - 1 - x)$ or $A(x)(1 - x - 2x^2) = \frac{3x^2}{1-x}$. Hence $A(x) = \frac{3x^2}{(1-x)(1-2x)(1+x)} = \frac{1}{2}\frac{1}{1+x} + \frac{1}{1-2x} - \frac{3}{2}\frac{1}{1-x} = \frac{1}{2}\sum_{n\geq 0}(-1)^nx^n + \sum_{n\geq 0}2^nx^n - \frac{3}{2}\sum_{n\geq 0}x^n$, so $a_n = \frac{(-1)^n-3}{2} + 2^n$.

9. Count the number of solutions to $a+b+c=n$ in nonnegative integers a, b, c, such that a is a multiple of 3, $b \leq 2$, and $c \geq 1$. Find a closed form for the sequence, and compute explicitly the value for $n=20$.

Call the desired answers g_n, and the generating function $G(x)=\sum_{n\geq 0} g_n x^n$. We have $G(x)=(1+x^3+x^6+x^9+\cdots)(1+x+x^2)(x+x^2+x^3+\cdots)=\frac{1}{1-x^3}\frac{1-x^3}{1-x}\frac{x}{1-x}=\frac{x}{(1-x)^2}=\sum_{n\geq 0} nx^n$. Hence $g_n=n$, and $g_{20}=20$.

10. Count the number of solutions to $a+b+c=n$ in nonnegative integers a, b, c, such that a is even, $b \leq 4$, and $c \geq 1$. Find a closed form for the sequence, and compute explicitly the value for $n=20$.

Call the desired answers g_n, and the generating function $G(x)=\sum_{n\geq 0} g_n x^n$. We have $G(x)=(1+x^2+x^4+\cdots)(1+x+x^2+x^3+x^4)(x+x^2+x^3+\cdots)=\frac{1}{1-x^2}(1+x+x^2+x^3+x^4)\frac{x}{1-x}=\frac{x+x^2+x^3+x^4+x^5}{(1-x^2)(1-x)}=x^2+2x+4-\frac{25}{4}\frac{1}{1-x}-\frac{1}{4}\frac{1}{1+x}+\frac{10}{4}\frac{1}{(1-x)^2}=x^2+2x+4-\frac{25}{4}\sum_{n\geq 0} x^n-\frac{1}{4}(-1)^n x^n+\frac{10}{4}\sum_{n\geq 0}(n+1)x^n=x^2+2x+4+\frac{1}{4}\sum_{n\geq 0}(-25-(-1)^n+10n+10)x^n$. Hence $g_n=\frac{10n-15-(-1)^n}{4}$ for $n\geq 3$. For $n=2$, we have $g_2=\frac{10\cdot 2-15-(-1)^2}{4}+1=2$, For $n=1$, we have $g_1=\frac{10\cdot 1-15-(-1)^1}{4}+2=1$, and lastly for $n=0$, we have $g_0=\frac{10\cdot 0-15-(-1)^0}{4}+4=0$. We also compute $g_{20}=46$.

11. Find the generating function for how many ways there are of making n cents in change, out of pennies, nickels, dimes, and quarters. Then compute explicitly the value for $n=111$.

Call the desired answers g_n, and the generating function $G(x)=\sum_{n\geq 0} g_n x^n$. We have $G(x)=(1+x+x^2+x^3+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+x^{30}+\cdots)(1+x^{25}+x^{50}+x^{75}+\cdots)=\frac{1}{1-x}\frac{1}{1-x^5}\frac{1}{1-x^{10}}\frac{1}{1-x^{25}}$. Don't even try to get the closed form for g_n, as most of the roots are complex and messy. Instead, just ask a computer (like Wolfram Alpha) to compute the power series for this function, out to the x^{111} term, which has coefficient 307. Hence $g_{111}=307$.

12. Consider the recurrence given by $c_0=1$, $c_{n+1}=\sum_{i=0}^{n} c_i$ $(n\geq 0)$. Find a generating function and a closed form for the sequence. Hint: Consider $\frac{C(x)}{1-x}$.

Following the hint, we have $C(x)\frac{1}{1-x}=\sum_{n\geq 0} c_n x^n \sum_{n\geq 0} x^n=\sum_{n\geq 0}(\sum_{i=0}^{n} c_i 1^{n-i})x^n=\sum_{n\geq 0}(\sum_{i=0}^{n} c_i)x^n=\sum_{n\geq 0} c_{n+1}x^n$. Multiplying both sides of this expression by x, we have $C(x)\frac{x}{1-x}=\sum_{n\geq 0} c_{n+1}x^{n+1}=C(x)-c_0=C(x)-1$. Rearranging, we get $C(x)(\frac{x}{1-x}-1)=-1$, or $C(x)=\frac{1-x}{1-2x}=\frac{1}{2}+\frac{1}{2}\frac{1}{1-2x}=\frac{1}{2}+\frac{1}{2}\sum_{n\geq 0} 2^n x^n$. Hence $c_n=2^{n-1}$, for $n\geq 1$. For $n=0$, we have $c_0=\frac{1}{2}+2^{n-1}=1$.