MATH 579: Combinatorics
Homework 6 Solutions

1. Prove the following properties for arbitrary constant C and functions $f(x), g(x)$.
   (a) $\Delta C = 0$;
   (b) $\Delta(Cf(x)) = C\Delta f(x)$; and
   (c) $\Delta(f(x)+g(x)) = (\Delta f(x)) + (\Delta g(x))$.

   We calculate $\Delta C = C - C = 0$, $\Delta(Cf(x)) = Cf(x+1) - Cf(x) = C(f(x+1) - f(x)) = C\Delta f(x)$, and $\Delta(f(x)+g(x)) = (f(x+1)+g(x+1)) - (f(x)+g(x)) = (f(x+1)-f(x)) + (g(x+1)-g(x)) = \Delta f(x) + \Delta g(x)$.

2. Find all functions $f(x)$ satisfying $\Delta(\Delta f(x)) = 3$.

   First, we find the functions $g(x)$ satisfying $\Delta g(x) = 3$. There are infinitely many, namely $g(x) = 3x^{\underline{1}} + C$, for any constant C. We now find the functions $f(x)$ satisfying $\Delta f(x) = g(x) = 3x^{\underline{1}} + C$. We get $f(x) = \frac{3}{2}x^{\underline{2}} + Cx^{\underline{1}} + D$.

3. Compute $\sum_{i=1}^{n} i^5$, for arbitrary $n \in \mathbb{N}$.

   We have $\displaystyle\sum_{i=1}^{n} i^5 = \sum_{1}^{n+1} x^5 \delta x = \sum_{1}^{n+1} x^{\underline{1}} + 15x^{\underline{2}} + 25x^{\underline{3}} + 10x^{\underline{4}} + x^{\underline{5}} \delta x$, using our table for $S(n,k)$.
   We continue as $\frac{1}{2}x^{\underline{2}} + 5x^{\underline{3}} + \frac{25}{4}x^{\underline{4}} + 2x^{\underline{5}} + \frac{1}{6}x^{\underline{6}}|_1^{n+1} = \frac{1}{2}(n+1)^{\underline{2}} + 5(n+1)^{\underline{3}} + \frac{25}{4}(n+1)^{\underline{4}} + 2(n+1)^{\underline{5}} + \frac{1}{6}(n+1)^{\underline{6}} - 0$.

4. Let $c \in \mathbb{R}$. Compute Δc^x. Use this to find an anti-difference of c^x, and hence the geometric sum $\sum_a^b c^x \delta x$ (for $c \neq 1$).

   We have $\Delta c^x = c^{x+1} - c^x = (c-1)c^x$, so an anti-difference is $\frac{c^x}{c-1}$ (for $c \neq 1$). Hence $\sum_a^b c^x \delta x = \frac{c^x}{c-1}|_a^b = \frac{c^b - c^a}{c-1}$.

5. For $c \in \mathbb{R}$ and $x \in \mathbb{N}$, compute $\Delta c^{\underline{x}}$. Use this to find an anti-difference of $\frac{(-2)^{\underline{k}}}{k}$, and hence the sum $\sum_{k=2}^{n} \frac{(-2)^{\underline{k}}}{k}$.

   We have $\Delta c^{\underline{x}} = c^{\underline{x+1}} - c^{\underline{x}} = c^{\underline{x}}(c-x-1) = c^{\underline{x+2}}/(c-x)$. Hence $\Delta(-2)^{\underline{x}} = (-2)^{\underline{x+2}}/(-2-x)$, and $\Delta(-2)^{\underline{x-2}} = (-2)^{\underline{x}}/(-2-(x-2)) = -(-2)^{\underline{x}}/x$. Taking negatives, we get $\Delta[-(-2)^{\underline{x-2}}] = (-2)^{\underline{x}}/x$. Hence the desired sum is $-(-2)^{\underline{x-2}}|_2^{n+1} = -(-2)^{\underline{n-1}} + (-2)^{\underline{0}} = 1 - (-2)^{\underline{n-1}}$.

6. For $k \in \mathbb{N}$, we define $x^{\underline{-k}} = \frac{1}{(x+1)(x+2)\cdots(x+k)}$. Prove that $\Delta x^{\underline{-k}} = -kx^{\underline{-k-1}}$.

   We calculate $\Delta x^{\underline{-k}} = (x+1)^{\underline{-k}} - x^{\underline{-k}} = \frac{1}{(x+2)(x+3)\cdots(x+k+1)} - \frac{1}{(x+1)(x+2)\cdots(x+k)} =$
   $= \frac{x+1}{(x+1)(x+2)(x+3)\cdots(x+k+1)} - \frac{x+k+1}{(x+1)(x+2)\cdots(x+k)(x+k+1)} = \frac{-k}{(x+1)(x+2)\cdots(x+k)(x+k+1)} = -kx^{\underline{-k-1}}$.

7. For $x \in \mathbb{N}$, we define $H_x = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}$. Prove that $\Delta H_x = x^{\underline{-1}}$.

   We calculate $\Delta H_x = H_{x+1} - H_x = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x} + \frac{1}{x+1} - (\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{x}) = \frac{1}{x+1} = x^{\underline{-1}}$.

8. Prove that $x^{\underline{m+n}} = x^{\underline{m}}(x-m)^{\underline{n}}$ for all integers m, n. (there are cases)

   Case 1: $m, n \geq 0$. (done in class)
   $x^{\underline{m}}(x-m)^{\underline{n}} = x(x-1)\cdots(x-m+1)(x-m)(x-m-1)\cdots(x-m-n+1) = x^{\underline{m+n}}$.

Case 2: $m, n < 0$. $x^{\underline{m}}(x-m)^{\underline{n}} = \frac{1}{(x+1)(x+2)\cdots(x-m)} \frac{1}{(x-m+1)(x-m+2)\cdots(x-m-n)} = x^{\underline{m+n}}$.

Case 3: $m \geq 0 > n$. $x^{\underline{m}}(x-m)^{\underline{n}} = x(x-1)\cdots(x-m+1)\frac{1}{(x-m+1)(x-m+2)\cdots(x-m-n)} = \frac{(x-m+1)(x-m+2)\cdots(x-m+m)}{(x-m+1)(x-m+2)\cdots(x-m-n)}$. Note that the terms in the numerator and denominator cancel, until they run out. If $m \geq -n$ (i.e. $m+n \geq 0$), then the result is $(x-m-n+1)\cdots(x-m+m) = x(x-1)\cdots(x-m-n+1) = x^{\underline{m+n}}$. If instead $m < -n$ (i.e. $m+n < 0$), then the result is $\frac{1}{(x-m+m+1)(x-m+m+2)\cdots(x-m-n)} = \frac{1}{(x+1)(x+2)\cdots(x-m-n)} = x^{\underline{m+n}}$.

Case 4: $n \geq 0 > m$. $x^{\underline{m}}(x-m)^{\underline{n}} = \frac{1}{(x+1)(x+2)\cdots(x-m)}(x-m)(x-m-1)\cdots(x-m-n+1) = = \frac{(x-m)(x-m-1)\cdots(x-m-n+1)}{(x-m)(x-m-1)\cdots(x-m+m+1)}$. Again the terms cancel nicely. If $n \geq -m$, then the result is $x(x-1)\cdots(x-m-n+1) = x^{\underline{m+n}}$. If instead $n < -m$, then the result is $\frac{1}{(x-m-n)(x-m-n-1)\cdots(x+1)} = \frac{1}{(x+1)(x+2)\cdots(x-m-n)} = x^{\underline{m+n}}$.

9. Calculate $\sum_0^n x3^x \delta x$. Your answer should be a function of n.

We set $u = x = x^{\underline{1}}, \Delta v = 3^x$. Note that $\Delta u = 1$ and $v = \frac{1}{2} \cdot 3^x$. We sum by parts, getting $\sum x3^x\delta x = x(\frac{1}{2})3^x - \sum \frac{1}{2}3^{x+1}\delta x = x(\frac{1}{2})3^x - \frac{3}{2}\sum 3^x \delta x = x(\frac{1}{2})3^x - \frac{3}{4}3^x$. Evaluating from 0 to n we get $n(\frac{1}{2})3^n - \frac{3}{4}3^n - (0 - \frac{3}{4}) = \frac{2n3^n - 3^{n+1}+3}{4}$.

10. Calculate $\sum_0^n x^2 2^x \delta x$.

We set $u = x^2 = x^{\underline{2}} + x^{\underline{1}}, \Delta v = 2^x = v$. Note that $\Delta u = 2x^{\underline{1}} + 1$. We sum by parts, getting $\sum x^2 2^x \delta x = x^2 2^x - \sum 2^{x+1}(2x^{\underline{1}}+1)\delta x = x^2 2^x - 4\sum x^{\underline{1}}2^x\delta x - 2\sum 2^x \delta x = x^2 2^x - 2^{x+1} - 4\sum x^{\underline{1}}2^x \delta x$. We sum by parts again, setting $u = x = x^{\underline{1}}, \Delta v = 2^x = v$, with $\Delta u = 1$. We get $\sum x^{\underline{1}}2^x\delta x = x2^x - \sum 2^{x+1}\delta x = x2^x - 2^{x+1}$. Combining, we get $\sum x^2 2^x \delta x = x^2 2^x - 2^{x+1} - 4(x2^x - 2^{x+1}) = x^2 2^x - x2^{x+2} + 3 \cdot 2^{x+1}$. Evaluating from 0 to n we get $n^2 2^n - n2^{n+2} + 3 \cdot 2^{n+1} - (0 - 0 + 6) = n^2 2^n - n2^{n+2} + 3 \cdot 2^{n+1} - 6$.

11. Calculate $\sum_0^n xH_x \delta x$. (hint: summation by parts and exercise 8)

We set $u = H_x$, $\Delta v = x = x^{\underline{1}}$. This gives $\Delta u = x^{\underline{-1}}$ and $v = \frac{1}{2}x^{\underline{2}}$. We sum by parts, getting $\sum xH_x\delta x = \frac{1}{2}x^{\underline{2}}H_x - \sum \frac{1}{2}(x+1)^{\underline{2}}x^{\underline{-1}}\delta x$. By Exercise 8, $(x+1)^{\underline{2}}x^{\underline{-1}} = x^{\underline{1}}$, so $\sum xH_x\delta x = \frac{1}{2}x^{\underline{2}}H_x - \frac{1}{2}\sum x^{\underline{1}}\delta x = \frac{1}{2}x^{\underline{2}}H_x - \frac{1}{4}x^{\underline{2}}$. Evaluating from 0 to n we get $\frac{1}{2}n^{\underline{2}}H_n - \frac{1}{4}n^{\underline{2}} - (0-0)$.

12. Calculate $\sum_1^n \frac{2x+1}{x(x+1)}\delta x$.

Solution 1: Breaking the fraction up, we get $\frac{2x+1}{x(x+1)} = \frac{2}{x+1} + \frac{1}{x(x+1)}$. Hence our sum is $\sum_1^n 2x^{\underline{-1}}\delta x + \sum_1^n (x-1)^{\underline{-2}}\delta x = \sum_1^n 2x^{\underline{-1}}\delta x + \sum_0^{n-1} x^{\underline{-2}}\delta x = 2H_x|_1^n - x^{\underline{-1}}|_0^{n-1} = 2H_n - 2H_1 - (n-1)^{\underline{-1}} + 0^{\underline{-1}} = 2H_n - 2 - \frac{1}{(n-1)+1} + \frac{1}{0+1} = 2H_n - \frac{1}{n} - 1$.

Solution 2: By partial fractions, we see that $\frac{2x+1}{x(x+1)} = \frac{1}{x} + \frac{1}{x+1}$. Hence our sum is $\sum_{k=1}^{n-1} \frac{1}{k} + \frac{1}{k+1} = \sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=1}^{n-1}\frac{1}{k+1} = H_{n-1} + (H_n - 1) = 2H_n - \frac{1}{n} - 1$.

Note: The original problem had a typo: it was missing δx, so full credit was given for either the above solution, or for solving the very similar problem $\sum_{x=1}^n \frac{2x+1}{x(x+1)}$, with solution $H_n + (H_{n+1} - 1) = 2H_{n+1} - \frac{1}{n+1} - 1$.