MATH 579: Combinatorics
Homework 5 Solutions

1. (Symmetry) \( \binom{a+b}{a} = \binom{a+b}{b} \).

Since \( a+b, a \in \mathbb{Z} \) with \( a+b \geq a \geq 0 \), we may use the factorial form of the binomial coefficient.
\[
\binom{a+b}{a} = \frac{(a+b)!}{a!(a+b-a)!} = \frac{(a+b)!}{a!b!} = \frac{(a+b)!}{b!(a+b-b)!} = \binom{a+b}{b}.
\]

2. (Pascal’s Rule) \( \binom{x}{a} + \binom{x}{a+1} = \binom{x+1}{a+1} \).

We calculate \( (a+1)x^a + x^{a+1} = x^a((a+1) + (x - (a+1) + 1)) = x^a(x+1) = (x+1)^{a+1} \). Divide both sides by \( (a+1)! \) and the result follows.

3. (Extraction) \( \binom{x}{a} = \frac{x(x-1) \cdots (x-a+1)}{a!} \). (provided \( a \neq 0 \))

Peeling off the first term, we see that \( x^a = x \cdot (x-1)^{a-1} \). Divide both sides by \( a! = a \cdot (a-1)! \) and the result follows.

4. (Committee/Chair) \( (a+1)\binom{x}{a+1} = x\binom{x-1}{a} \).

This symmetric version of the extraction identity comes from multiplying both sides by \( a \), and replacing \( a \) by \( a+1 \). It gets its name from the special case when \( x \in \mathbb{N} \). Then, the LHS counts the ways to pick a committee of \( a+1 \) out of \( x \) people, then pick a chair from the committee’s members. The RHS counts the ways to pick the chair first, out of \( x \) people, then pick the remaining \( a \) members of the committee out of the remaining \( x-1 \) people.

5. (Twisting) \( \binom{x}{a} \binom{x-a}{b} = \binom{x}{b} \binom{x-a}{a} \).

We see that \( x^a(x-a)^b = x(x-1) \cdots (x-a+1)(x-a)(x-a-1) \cdots (x-a-b+1) = x^{a+b} = x(x-1) \cdots (x-b+1)(x-b)(x-b-1) \cdots (x-b-a+1) = x^a(x-b)^a \). Divide both sides by \( ab! = b!a! \) and the result follows.

6. (Negation) \( \binom{x}{a} = (-1)^a \binom{-x-1}{a} \).

We write \( x^a = (x-0)(x-1)(x-2) \cdots (x-a+2)(x-a+1) = (-1)^a(0-x)(1-x)(2-x) \cdots (a-x-2)(a-x-1)(a-x-2) \cdots (a-x-1)(a-x-2)(a-x-3) \cdots (a-x-1-a+1)(a-x-1)(a-x-1-a+1) = (-1)^a(a-x-1)^a \).

Divide both sides by \( a! \) and the result follows.

7. \( \binom{-1}{a} = (-1)^a \binom{2a}{a} \cdot 2^{-2a} \).

For this problem and the next it is useful (but not necessary) to define the double factorial, \( n!! = n \cdot (n-2) \cdots 4 \cdot 2 \), with \( 0!! = 1!! = 1 \). We now prove a lemma: For \( n = 2k - 1 \) odd, \( n!! = (2k-1)! \).

Proof: Induction on \( k \). \( k = 1 \) !!!! = 1 = \( \frac{2!}{2!} \). Assume that \( n!! = \frac{(2k)!}{2^kk!} \), and multiply both sides by \( n + 2 = 2k + 1 \). We get \( (n+2)!! = (n+2) \cdot n!! = \frac{(2k+1)(2k)!}{2^k(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{(2k+1)2^k(k+1)!} = \frac{(2k+1)!}{2^k(k+1)!} \).

Now, \( (-\frac{1}{2})^{a} = (\frac{1}{2})^{a-1} = (\frac{1}{2}) \cdots (\frac{1}{2} - a + 1) = (\frac{1}{2})(\frac{3}{2})(\frac{5}{2}) \cdots (\frac{2a-1}{2}) = (-1)^{a}2^{-a}(2a - 1)!! = (-1)^{a}2^{-a}(2a - 1)!! = (-1)^{a}2^{-a}(2a - 1)!! \cdot 2^{-2a} = (-1)^{a}2^{-a}(2a - 1)!! \cdot 2^{-2a} = (2a - 1)!! \cdot 2^{-2a} \).

Now divide both sides by \( a! \).

8. \( \binom{\frac{1}{2}}{a} = (-1)^{a+1}(\frac{2a}{a}) \cdot 2^{-2a} \).

We have \( \binom{\frac{1}{2}}{a} = (\frac{1}{2})(\frac{1}{2} - 1)(\frac{1}{2} - 2) \cdots (\frac{1}{2} - a + 1) = (\frac{1}{2})(\frac{3}{2})(\frac{5}{2}) \cdots (\frac{2a-1}{2}) = (-1)^{a+1}(\frac{2a}{a}) \cdot 2^{-2a} \).

Now divide both sides by \( a! \).
9. \[(\text{Chu-Vandermonde}) \quad \binom{x+y}{a} = \sum_{k=0}^{a} \binom{x}{k} \binom{y}{a-k}. \quad \text{Hint: } (t+1)^x(t+1)^y\]

Assuming \(|t| < 1\), we apply Newton’s binomial theorem three times as follows. \[
\sum_{a \geq 0} \binom{x+y}{a} t^a = (t+1)^{x+y} = (t+1)^x(t+1)^y = \left( \sum_{a \geq 0} \binom{x}{a} t^a \right) \left( \sum_{a \geq 0} \binom{y}{a} t^a \right) = \sum_{a \geq 0} \left( \sum_{k=0}^{a} \binom{a}{k} \binom{y}{a-k} \right) t^a,
\] using the formula for the product of power series. We now equate coefficients of \(t^a\) and are done.

10. \[(\text{Chu-Vandermonde II}) \quad (x+y)^a = \sum_{k=0}^{a} \binom{a}{k} x^k y^{a-k}.\]

Multiply both sides of the Chu-Vandermonde identity by \(a!\) and note that \(a! \binom{x}{k} \binom{y}{a-k} = \frac{a!}{k!(a-k)!} x^k y^{a-k} = \binom{a}{k} x^k y^{a-k}.

11. \[
\sum_{k=0}^{a} \binom{a}{k}^2 = \binom{2a}{a}.\quad \text{Hint: Chu-Vandermonde}
\]

Apply Chu-Vandermonde with \(x = y = a\). Note that, by the symmetry identity, \(\binom{a}{a-k} = \binom{a}{k}\).

12. \[(\text{Hockey Stick}) \quad \sum_{k=a}^{a+b} \binom{k}{a} = \binom{a+b+1}{a+1}.\]

Induction on \(b\). If \(b = 0\), the LHS is \(\binom{a}{a} = 1 = \binom{a+1}{a+1}\). Suppose now that \(\sum_{k=a}^{a+b} \binom{k}{a} = \binom{a+b+1}{a+1}\), and add \(\binom{a+b+1}{a}\) to both sides. We have \(\sum_{k=a}^{a+b+1} \binom{k}{a} = \binom{a+b+1}{a} + \binom{a+b+1}{a+1} = \binom{a+b+2}{a+1}\), applying Pascal’s Rule.

13. Suppose that \(b \leq \frac{a-1}{2}\). Then \(\binom{a}{b} \leq \binom{a}{b+1}\).

We have \(a \geq 2b+1\), hence \(a-b \geq b+1\), hence \(\frac{1}{b+1} \leq \frac{1}{a-b}\). We multiply both sides by \(\frac{a!}{b!(a-b-1)!}\) to get \(\frac{a!}{(b+1)!(a-b-1)!} \geq \frac{a!}{b!(a-b)!}\), the desired result.

14. Suppose that \(b \geq \frac{a-1}{2}\). Then \(\binom{a}{b} \geq \binom{a}{b+1}\).

Set \(b' = a - (b+1)\). Since \(b \geq \frac{a-1}{2}\), \(b+1 \geq \frac{a+1}{2}\) and hence \(b' = a - (b+1) \leq a - \frac{a+1}{2} = \frac{a-1}{2}\). Apply the previous problem to get \(\binom{a}{b'} \leq \binom{a}{b+1}\). Apply the symmetry identity twice to get \(\binom{a}{a-b'} \leq \binom{a}{a-b+1}\), which is the desired result since \(a-b' = b+1\) and \(a-b' - 1 = b\).

This problem, and the previous, prove that each row of Pascal’s triangle is nondecreasing until the middle, and then nonincreasing. Such sequences are called unimodal.

15. \(\frac{2^n}{2n+1} \leq \binom{2n}{n} \leq 4^n.\quad \text{Hint: } (1+1)^{2n}\)

We have \(4^n = (1+1)^{2n} = \sum_{i=0}^{2n} \binom{2n}{i}\), by Newton’s binomial theorem. Since all the summands are nonnegative, if we replace all but \(\binom{2n}{n}\) with zero, the sum only decreases: \(4^n = \sum_{i=0}^{2n} \binom{2n}{i} \geq \binom{2n}{n}\). This gives the upper bound. By unimodality proved by the previous two problems, the largest summand is \(\binom{2n}{n}\). Hence if we replace each summand by this largest one, the sum only increases: \(4^n = \sum_{i=0}^{2n} \binom{2n}{i} \leq (2n+1)\binom{2n}{n}\). Dividing by \(2n+1\) gives the lower bound.