1. Use difference calculus to compute $\sum_{i=1}^{100} i^4$.

We have $\sum_{i=1}^{100} i^4 = \sum_{i=1}^{101} i^4 \delta x = \sum_{i=1}^{101} x^4 \delta x = \frac{1}{2} x^2 + \frac{2}{3} x^2 + \frac{6}{4} x^2 + \frac{1}{5} x^2 \delta x$, using our table for $S(n,k)$.

We continue as $\frac{1}{2} x^2 + \frac{2}{3} x^2 + \frac{6}{4} x^2 + \frac{1}{5} x^2 \delta x = \frac{1}{2}(101)^2 + \frac{2}{3}(101)^2 + \frac{6}{4}(101)^2 + \frac{1}{5}(101)^2 = 2,050,333,330$.

2. Let $n \in \mathbb{N}$. Prove that $n(\binom{2n-1}{n-1}) = \sum_{k=0}^{n} k(n)_k^2$.

By the committee/chair identity, $k(n)_k = n(n-1)_k$. By symmetry, $n(n-1)_k = n(n-1)_k$. Hence

$\sum_{k=0}^{n} k(n)_k^2 = \sum_{k=0}^{n} n(n-1)_k(n)_k(n)_k(n-1)_k = n(n-1)_2(n)_n(n-1)_n = n(\binom{2n-1}{n})$. The last step follows from the Chu-Vandermonde identity with $x = n, y = n-1, a = n$. By symmetry, $n(\binom{2n}{n}) = n(n-1)_2(n)_n(n-1)_n$.

3. Let $n \in \mathbb{N}$. Prove that $n(n+1)n^2 = \sum_{k=0}^{n} k^2(n)_k$.

We begin with the binomial theorem, stating $(x+1)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. Taking $\frac{d}{dx}$ of both sides, we get $n(x+1)^n = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1}$. Multiplying both sides by $x$, we get $nx(x+1)^n = \sum_{k=0}^{n} k \binom{n}{k} x^k$. Taking $\frac{d}{dx}$ of both sides, we get $n[(x+1)^n + (n-1)x(x+1)^{-2}] = \sum_{k=0}^{n} k^2 \binom{n}{k} x^{k-1}$. Taking $x = 1$, we get $\sum_{k=0}^{n} k^2 \binom{n}{k} = n(2^n + (n-1)2^{n-2}) = n(n+1)^2$.

4. Let $k \in \mathbb{Z}$, $x \in \mathbb{R}$ with $x > k - 2 \geq 0$. Prove that $\binom{x+2}{k} k^2 \binom{x}{k} \leq \binom{x-1}{k}^2$.

Since $k \geq 2, -2k \leq -k$, and hence $x^2 + 3x - kx - 2k + 2 \leq x^2 + 3x - kx - k + 2$. We factor each side to get $(x-k)(x+2) \leq (x-k+2)(x+1)$. Since $x > k-2, x-k+2$ is positive (and so is $x+1$). Hence we can divide by the positive RHS to conclude $\binom{x-1}{k}^2 \binom{x-2}{k} \leq \binom{x-2}{k} \binom{x-1}{k}^2 = \binom{x-1}{k}^2$.

5. Let $n, k \in \mathbb{Z}$ with $n > 1$ and $k > 1$. Prove that $k^n < \binom{n}{n}^k$.

We compute $\binom{n}{n}^k = \binom{n-1}{n-1}^k = \binom{n-k}{n-1}^k = \binom{n-k-1}{n-1}^k = \binom{n-k-2}{n-1}^k \cdots \binom{n-k-i}{n-1}^k$, which we can write as $\binom{n-1}{n-1} \binom{n-2}{n-1} \cdots \binom{n-k-i}{n-1}^k$. Now, $k > 1$ so for $i \geq 1$ we have $ki > i$, which rearranges to $nk-ki > n(k-i)$. Since $i \leq n-1$, we divide by the positive $n-i$ to get $\frac{nk-i}{n-i} > k$. For $i = 0, n\frac{k-i}{n-i} = k$. Hence $\binom{n}{n}^k > k^n$ for $n > 1$, and for $n > 1$ the inequality is strict.

6. Compute $\sum_{k=1}^{n} \frac{H_k}{(k+1)(k+2)}$.

We write $\sum_{k=1}^{n} \frac{H_k}{(k+1)(k+2)} = \sum_{k=1}^{n+1} H_k x^2 \delta x$. We set $u = H_x, \Delta v = x^2$. This gives $\Delta u = x^{-1}$ and $v = -x^{-1}$. We sum by parts, getting $\sum H_x x^2 \delta x = -x^{-1} H_x - \sum -(x+1)^{-1} x^{-1} \delta x = -x^{-1} H_x + \sum x^{-2} \delta x = -x^{-1} H_x - x^{-1} = -x^{-1} (H_x + 1)$. We evaluate from 1 to $n+1$, getting $-(n+1)^{-1} (H_{n+1} + 1) + 1^{-1} (H_1 + 1) = -\frac{H_{n+1} + 1}{n+2} + 1$. Note that as $n \to \infty$, the fraction approaches 0, so the sum approaches 1.