MATH 579 Final Exam; 12/12/13

Please read the exam instructions.

Please indicate what work goes with which problem and put your name or initials on every sheet. Cross out work you do not wish graded; incorrect work can lower your grade, even compared with no work at all. Show all necessary work in your solutions; if you are unsure, show it. Simplify all numerical answers to be integers, if possible. You are welcome to use your book, notes, and calculators; if you use an earlier result be sure to cite it. You have 120 minutes. This exam is out of 60 points maximum.

Part I: Do all three problems.

For these three problems we call $n \in \mathbb{N}$ *comfortable* if it satisfies the following property:

If prime p divides n, then p^3 divides n.

For example, $1, 8, 16, 27, 216\ (2^3 3^3), 648\ (2^3 3^4)$ are each comfortable, while $2, 4, 6, 24$ are not.

1. (5-10 points) How many comfortable divisors does $270,000$ have?
2. (5-10 points) How many (positive) divisors of 60^8 are neither comfortable nor square?
3. (5-10 points) Let n be comfortable, with prime decomposition $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$. Let $d(n)$ denote the number of (positive) divisors of n, and $c(n)$ denote the number of comfortable divisors of n. Prove that $c(n) \geq \frac{d(n)}{2^k}$, and characterize those n where equality holds.

Part II: Choose three of the following six problems.

4. (5-8 points) How many permutations $p \in S_7$ satisfy $p^3 = 1$?
5. (5-10 points) Calculate how many solutions there are to $a+b+c = 101$ such that a, b, c are nonnegative integers and $a \leq 21$.
6. (5-10 points) For $n \in \mathbb{N}$, simplify $\binom{-\frac{5}{2}}{n}$ to an expression containing only factorials and exponentials.
7. (5-10 points) Consider the sequence given by $a_0 = -1, a_1 = 2, a_n = 2a_{n-1} - a_{n-2}$ $(n \geq 2)$. Find a closed form for a_n.
8. (5-10 points) Consider the sequence given by $a_0 = -1, a_n = na_{n-1} + n!$ $(n \geq 1)$. Find a closed form for a_n.
9. (5-12 points) Consider the sequence given by $a_0 = 1, a_{n+1} = 2\sum_{i=0}^{n} a_i a_{n-i}$ $(n \geq 0)$. Find a closed form for a_n.

Extra credit: Predict your score on this exam, out of 60. If close (within 2 points), you'll earn a bonus point. If exactly right, you'll earn two bonus points.