Please indicate what work goes with which problem and put your name or initials on every sheet. Cross out work you do not wish graded; incorrect work can lower your grade, even compared with no work at all. Show all necessary work in your solutions; if you are unsure, show it. Simplify all numerical answers to be integers, if possible. You are welcome to use your book, notes, and calculators; if you use an earlier result be sure to cite it. You have 120 minutes. This exam is out of 60 points maximum.

Part I: Do all three problems.

For these three problems we call \( n \in \mathbb{N} \) comfortable if it satisfies the following property:

If prime \( p \) divides \( n \), then \( p^3 \) divides \( n \).

For example, 1, 8, 16, 27, 216 \((2^33^3)\), 648 \((2^33^4)\) are each comfortable, while 2, 4, 6, 24 are not.

1. (5-10 points) How many comfortable divisors does 270,000 have?
2. (5-10 points) How many (positive) divisors of 60\(^8\) are neither comfortable nor square?
3. (5-10 points) Let \( n \) be comfortable, with prime decomposition \( n = p_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} \). Let \( d(n) \) denote the number of (positive) divisors of \( n \), and \( c(n) \) denote the number of comfortable divisors of \( n \). Prove that \( c(n) \geq \frac{d(n)}{2^k} \), and characterize those \( n \) where equality holds.

Part II: Choose three of the following six problems.

4. (5-8 points) How many permutations \( p \in S_7 \) satisfy \( p^3 = 1? \)
5. (5-10 points) Calculate how many solutions there are to \( a + b + c = 101 \) such that \( a, b, c \) are nonnegative integers and \( a \leq 21 \).
6. (5-10 points) For \( n \in \mathbb{N} \), simplify \( \left( -\frac{5}{2} \right)^n \) to an expression containing only factorials and exponentials.
7. (5-10 points) Consider the sequence given by \( a_0 = -1, a_1 = 2, a_n = 2a_{n-1} - a_{n-2} \) \((n \geq 2)\). Find a closed form for \( a_n \).
8. (5-10 points) Consider the sequence given by \( a_0 = -1, a_n = na_{n-1} + n! \) \((n \geq 1)\). Find a closed form for \( a_n \).
9. (5-12 points) Consider the sequence given by \( a_0 = 1, a_{n+1} = 2 \sum_{i=0}^{n} a_ia_{n-i} \) \((n \geq 0)\). Find a closed form for \( a_n \).

Extra credit: Predict your score on this exam, out of 60. If close (within 2 points), you’ll earn a bonus point. If exactly right, you’ll earn two bonus points.