Math 579 Fall 2013 Exam 9 Solutions

1. Describe a process that yields a sequence whose generating function is given by $\frac{1}{(1-x)^2} = \frac{(1-x)^2}{x^2-3x+1}$.

This is Example 8.14 in the text. Given a list of $m$ objects, there are $m$ ways to pick one distinguished object. This has generating function $\frac{x}{(1-x)^2} = \sum_{m \geq 0} mx^m$. Now, we divide a list of $n$ objects into nonempty sublists. From each sublist we choose one element to distinguish. We now use Thm 8.13.

2. Let $k \in \mathbb{N}_0$, and consider the sequence given by $a_n = \binom{n+k}{k}$. Prove that it has generating function $A(x) = \frac{1}{(1-x)^{k+1}}$.

We use induction on $k$. For $k = 0$ we have the geometric series $\frac{1}{1-x} = \sum_{n \geq 0} x^n$. Suppose now that $\frac{1}{1-x} = \sum_{n \geq 0} x^n$. We take derivatives of each side to get $(-k)(1-x)^{-k-1} = \sum_{n \geq 0} (n+1)k x^{n-1}$. We drop the initial zero term, then reindex to get $\frac{k}{(1-x)^{k+1}} = \sum_{n \geq 0} \binom{n+k}{k}(n+1)x^n$. We divide both sides by $k$, and note that $\binom{n+k}{k} = \frac{(n+k)!}{(n+1)!k!} = \frac{(n+k)!}{n!k!} = \binom{n+k}{k}$.

3. Consider the sequence $a_n = n^2 + 2n + 3 + (-1)^n$ for $n \geq 0$. Find and simplify the generating function for this sequence.

We use the result of problem 2. We have $2\binom{n+2}{2} = (n+2)(n+1) = n^2 + 3n + 2$ We have $\binom{n+1}{k} = n+1$. Hence $n^2 + 2n + 3 = 2\binom{n+2}{2} - \binom{n+1}{2} + 2$, so $A(x) = \frac{2}{(1-x)^2} - \frac{1}{1-x} + \frac{2}{1+x} + \frac{1}{1+x} = \frac{x^2 + 2x - 3x + 3}{(1-x)^3(1+x)}$.

4. Use generating functions to solve the recurrence specified by $a_0 = 2, a_n = 3a_{n-1} + 2 (n \geq 1)$.

Let $A(x) = \sum_{n \geq 0} a_n x^n$. We have $A(x) - 3x A(x) - \frac{2}{1-x} = \sum_{n \geq 0} a_n x^n + \sum_{n \geq 0} (-3a_n) x^{n+1} + \sum_{n \geq 0} (-2)x^n = \sum_{n \geq 0} a_n x^n + \sum_{n \geq 1} (-3a_{n-1}) x^n + \sum_{n \geq 0} (-2)x^n = a_0 x^0 + (-2) x^0 + \sum_{n \geq 1} (a_n - 3a_{n-1} - 2)x^n = 0$. Hence $(1-3x)A(x) = \frac{2}{1-x} - \frac{2}{1+x}$, so $A(x) = \frac{2}{(1-x)(1-3x)} = \frac{3}{1-4x} + \frac{1}{1-x}$, where the last step required partial fractions.

Hence $A(x) = 3 \sum_{n \geq 0} 3^n x^n - \sum_{n \geq 0} 1x^n = \sum_{n \geq 0} (3^n + 1)x^n$, so $a_n = 3^{n+1} - 1$.

5. When faced with a stack of exams to grade, I give each exam in turn either a pass or a fail. At some point (perhaps before I even start), my booze runs out and I stop grading immediately. The last thing I do is choose one of the remaining exams (and there is always at least one exam left), on which I draw a smiley face. Let $a_n$ represent the number of ways I can do this with $n$ exams. For example, $a_0 = 0$ (can’t draw a smiley), $a_1 = 1$, $a_2 = 4$. Find a closed form for $a_n$.

Let $G(x) = \sum_{n \geq 0} 2^n x^n = \frac{1}{1-2x}$ denote the grading part, and $S(x) = \sum_{n \geq 0} nx^n = \frac{x}{(1-x)^2}$ denote the smiley part. We use these with the product formula $A(x) = G(x)S(x) = \frac{x}{(1-2x)(1-x)^2} = \frac{2}{1-2x} + \frac{1}{1-x} + \frac{1}{(1-x)^2}$, where partial fractions were used in the last step. We now have $A(x) = 2 \sum_{n \geq 0} 2^n x^n - \sum_{n \geq 0} 1x^n - \sum_{n \geq 0} (n+1)x^n = \sum_{n \geq 0} (2^{n+1} - 1 - (n+1))x^n$, so $a_n = 2^{n+1} - n - 2$. 