1. Calculate how many permutations of \([5]\) contain none of the cycles \((1\ 2), (2\ 3),\) or \((3\ 4)\).

Let \(S = \{s_1, s_2, s_3\}\) where \(s_i\) denotes that the cycle \((i\ i+1)\) is present. We calculate \(f_\geq(\emptyset) = 5! = 120,\) and \(f_\geq(s_i) = 3! = 6,\) since assuming any one of the cycles there are three remaining elements to permute. We have \(f_\geq(s_1 s_2) = f_\geq(s_2 s_3) = f_\geq(s_1 s_2 s_3) = f_\geq(s_1 s_3 s_2) = 0,\) since if \((2\ 3)\) is present neither of the other two cycles can be present. However \(f_\geq(s_1 s_3) = 1,\) specifically \((1\ 2)(3\ 4)(5)\). We want \(f_\geq = f_\geq(\emptyset) = 120 - 6 - 6 - 6 + 1 + 1 = 103.\)

2. Calculate how many permutations \(\pi\) of \([5]\) satisfy \(\pi(1) \neq 2, \pi(2) \neq 3, \pi(3) \neq 4, \pi(4) \neq 5.\)

The diagram at left indicates the forbidden positions. We calculate \(r_1 = 4, r_2 = \binom{4}{2} = 6, r_3 = \binom{4}{2} = 6, r_4 = \binom{4}{2} = 6.\) The formula we want is \(5! - 4! + 3! - 2! + 1! = 53.\)

3. Calculate how many permutations of \([5]\) have exactly one fixed point.

There are five possible fixed points, and the remaining elements form a derangement of four elements, of which there are \(D(4) = 9,\) so the answer is \(5 \times 9 = 45.\)

4. Calculate how many ways we can list the digits \(\{1, 1, 2, 2, 3, 3, 4\}\) so that two identical digits are not in consecutive positions.

Let \(S = \{s_1, s_2, s_3\}\) where \(s_i\) denotes that the two \(i's\) are consecutive. We calculate \(f_\geq(\emptyset) = \frac{7!}{2! 2! 2!} = 630,\) \(f_\geq(s_i) = \frac{6!}{2! 2!} = 180, f_\geq(s_i s_j) = \frac{5!}{2!} = 60,\) and \(f_\geq(s_1 s_2 s_3) = 4! = 24.\) We want \(f_\geq(\emptyset) = 630 - 3(180) + 3(60) - 24 = 246.\)

5. Calculate how many ways we can list the digits \(\{1, 1, 2, 2, 3, 3, 4\}\) so that two identical digits are not in consecutive positions.

Let’s write \(1_A, 1_B, 1_C\) to distinguish the 1’s; we will divide by 6 in the end. Let \(S = \{s_1, s_2, s_3, r\}\) where \(s_1\) denotes \(1_A\) and \(1_B\) together, \(s_2\) denotes \(1_A\) and \(1_C\) together, \(s_3\) denotes \(1_B\) and \(1_C\) together, \(r\) denotes the 2’s together. We calculate \(f_\geq(\emptyset) = \frac{7!}{2!} = 2520, f_\geq(s_i) = \frac{6!}{2! 2!} = 720 (2 \text{ because } 1_A1_B\) or \(1_B1_A),\) \(f_\geq(r) = 6! = 720, f_\geq(s_i s_j) = \frac{5!}{2!} = 120 (2 \text{ because } 1_A1_B1_C\) or \(1_C1_B1_A),\) \(f_\geq(s_i r) = 2(5!) = 240.\) We can’t have all three of \(s_1, s_2, s_3,\) but \(f_\geq(s_i s_j r) = 2(4!) = 48.\) Putting it all together, \(f_\geq(\emptyset) = 2520 - 3(720) - 720 + 3(120) + 3(240) - 3(48) = 576.\) Finally, we use equivalence classes to erase the subscripts, giving a final answer of \(\frac{576}{6} = 96.\)