Math 579 Fall 2013 Exam 2 Solutions

1. Prove that \( \sum_{i=1}^{n} i(i-3) = \frac{(n-4)n(n+1)}{3} \).

Proof by induction on \( n \), natch. Base case: \( n = 1 \), LHS=1(1-3)=-2 = \( \frac{(1-4)(1+1)}{3} \)=RHS.

Assume that \( \sum_{i=1}^{n} i(i-3) = \frac{(n-4)n(n+1)}{3} \) holds, and add \( (n+1)(n+1-3) \) to both sides. Then

\[
\sum_{i=1}^{n+1} i(i-3) = \frac{(n-4)n(n+1)}{3} + (n+1)(n-2) = (n+1)\left(\frac{n^2-4n}{3} + \frac{3n-6}{3}\right) = (n+1)\left(\frac{n^2-n-6}{3}\right) = (n+1)\frac{(n-2)(n+3)}{3},
\]

as desired.

2. Let \( a_1 = 1, a_2 = 5, \) and \( a_n = a_{n-1} + 2a_{n-2} \) for \( n \geq 2 \). Prove that \( a_n = 2^n + (-1)^n \).

Proof by strong induction on \( n \). Two base cases: \( n = 1 \) \( a_1 = 1 = 2^1 + (-1)^1 \) and \( a_2 = 5 = 2^2 + (-1)^2 \).

Now we have \( a_n = a_{n-1} + 2a_{n-2} \). Applying the inductive hypothesis to each summand, we get

\[
a_n = 2^{n-1} + (-1)^{n-1} + 2(2^{n-2} + (-1)^{n-2}) = 2^{n-1} + 2 \cdot 2^{n-2} + (-1)^{n-2}(1+2) = 2^n + (-1)^n,
\]

as desired.

3. Recall that \( F_i \) denotes the Fibonacci numbers, i.e. \( F_1 = F_2 = 1 \) and \( F_j + F_{j+1} = F_{j+2} \) for \( j \geq 1 \). Prove that \( \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \), for all \( n \in \mathbb{N} \).

Proof by induction on \( n \). Base case: \( n = 1 \), LHS\( = F_1^2 = 1 = F_1 F_2 \).

Assume now that \( \sum_{i=1}^{n} F_i^2 = F_n F_{n+1} \). Add \( F_{n+1}^2 \) to both sides, that gives \( \sum_{i=1}^{n+1} F_i^2 = F_n F_{n+1} + F_{n+1}^2 = F_{n+1}(F_n + F_{n+1}) = F_{n+1} F_{n+2} \), as desired.

4. Use induction to prove that \( \frac{d}{dx} x^n = nx^{n-1} \), for all \( n \in \mathbb{N} \).

Proof by induction on \( n \). Base case \( n = 1 \), we use the definition of derivative to find

\[
\frac{d}{dx} x^1 = \lim_{\epsilon \to 0} \frac{(x + \epsilon) - x}{\epsilon} = \lim_{\epsilon \to 0} 1 = 1.
\]

We now use the product rule as \( \frac{d}{dx} (x^n) = \frac{d}{dx} (x \cdot x^{n-1}) = 1 \cdot x^{n-1} + x \cdot (n-1)x^{n-2} = x^{n-1}(1+n-1) = x^{n-1}(n) \), where we used the inductive hypothesis to conclude that \( \frac{d}{dx} x^{n-1} = (n-1)x^{n-2} \).

5. A tree is a connected simple finite graph with no cycles. Prove that every tree on \( n \) vertices must have exactly \( n-1 \) edges. You may use freely the following result:

**Theorem:** Every tree with at least two vertices has at least two leaves.

Proof by induction on \( n \). Pre-base-case: If \( n = 1 \), then the tree has no edges, so the result is true. Base case: If \( n = 2 \), then since the tree is connected the two vertices must have an edge between them, so there is exactly one edge, so the result is true.

Inductive case: Consider an arbitrary tree on \( n \) vertices, and choose a leaf vertex whose existence is guaranteed by the theorem. Deleting this vertex and its lone attached edge gives a smaller graph. The resulting graph is still connected, simple, finite, and has no cycles, hence is a tree with \( n - 1 \) vertices. By the inductive hypothesis, it has \( n - 2 \) edges. Thus, the original tree must have had \( n - 1 \) edges.