Math 524 Exam 8 Solutions

All the problems concern the vector space \( \mathbb{R}_4 \) and the bilinear real symmetric form \( \langle f | g \rangle = \int_0^1 f(t)g(t)dt \).

1. Under the standard basis \( E = \{1, t, t^2\} \), find the metric \( G_E \).

\( G_E \) is the matrix satisfying \( \langle f | g \rangle = [f]_E^T G_E [g]_E \). Setting \( e_1 = 1, e_2 = t, e_3 = t^2 \), we need to compute \( \langle e_i | e_j \rangle \) for every \( i, j \). By symmetry, this will only be 6 integrals, and none of them are difficult. \( \langle e_1 | e_1 \rangle = \int_0^1 1 \, dt = 1 \), \( \langle e_1 | e_2 \rangle = \int_0^1 t \, dt = 1/2 \), \( \langle e_1 | e_3 \rangle = \int_0^1 t^2 \, dt = 1/3 \), \( \langle e_2 | e_2 \rangle = \int_0^1 t^3 \, dt = 1/4 \), \( \langle e_2 | e_3 \rangle = \int_0^1 t^4 \, dt = 1/5 \). Putting it all together, we get \( G_E = \begin{pmatrix} 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \\ 1/4 & 1/5 & 1/6 \end{pmatrix} \). Matrices with this particular structure are called Hilbert matrices.

2. Prove that the above form is a (real) inner product.

A real inner product requires five properties: linearity in each coordinate, symmetry, positivity, and definiteness. The first three are given for free (or are easy to check by properties of the integral). Definiteness is also easy, since \( \langle 0 | 0 \rangle = \int_0^1 0 \, dt = 0 \). The only significant issue is positivity.

Analytic solution: Suppose first that \( f^2(a) = b \), for some \( a \in (0, 1) \), \( b > 0 \). Then, because polynomials are continuous, there is some interval \( [a - \epsilon, a + \epsilon] \), in which \( f^2 \geq b/2 \). Hence, \( \langle f | f \rangle = \int_0^1 f^2(t) \, dt \geq \int_{a - \epsilon}^{a + \epsilon} (b/2) \, dt = \epsilon b/2 = \epsilon b > 0 \). Hence if \( f \) is nonzero at ANY point in \( (0, 1) \), then \( \langle f | f \rangle > 0 \). On the other hand, if \( f \) is zero at every point of \( (0, 1) \), then it must be the zero polynomial. [proof: by the fundamental theorem of algebra, the only polynomial with infinitely many roots is the zero polynomial].

Algebraic solution: The form is positive if the matrix \( G \) is positive definite. \( G \) is symmetric, so by Sylvester’s criterion we need only check three determinants. \( |1| = 1 \), \( \begin{vmatrix} 1/2 \\ 1/3 \\ 1/4 \end{vmatrix} = 1/12 \). All are positive, hence \( G \) is positive definite.

The last two problems refer to the vectors \( u(t) = t - 1, v(t) = t^2 - 1 \). Set \( V = \text{Span}(u, v) = \{at^2 + bt - (a + b) \} \).

3. Find an orthogonal basis for \( V \).

\( \{u, v\} \) is a basis already, but not an orthogonal one; hence Gram-Schmidt is in order. An orthogonal basis will be \( \{u, v\} \), for \( w = v - P_{ru}[v] = v - \frac{\langle u | v \rangle}{\langle u | u \rangle} u \). We calculate

\[ \langle u | v \rangle = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 & 0 \\ 1/3 & 1/3 \\ 1/3 & 1/4 \\ 1/4 & 1/5 \end{pmatrix} = 5/12 \],

\[ \langle u | u \rangle = \begin{pmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{pmatrix} = 1/4 \].

Hence \( w = v - 5/4u = t^2 - 1 - 1.25t + 0.25 \).

4. For basis \( B = \{u, v\} \), calculate two bases for \( V^* \) by specifying their action on each element of \( V \). (1) the dual basis \( \{\phi_1, \phi_2\} \), (2) the bra basis \( \{\langle u |, \langle v | \} \).

We have \( x(t) = at^2 + bt - (a + b) = au + bv \). Hence \( [x]_B = \begin{pmatrix} a \\ b \end{pmatrix} \), and \( \phi_1(x) = a, \phi_2(x) = b \).

\[ \langle u | x \rangle = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{pmatrix} = \frac{5a + 4b}{12} \],

\[ \langle v | x \rangle = \begin{pmatrix} -1 & 0 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \end{pmatrix} = \frac{-a - b}{12} \]