All problems are for the vector space $\mathbb{R}_2[t]$, real polynomials of degree at most 2. We define $V = \{ p(t) : p(1) = 0 \}$, a subspace of $\mathbb{R}_2[t]$.

1. Let $A = \{ a_1, a_2 \}$ for $a_1 = t - 1, a_2 = t^2 - 1$. Let $B = \{ b_1, b_2 \}$ for $b_1 = t^2 + t - 2, b_2 = t^2 + 2t - 3$. Prove that $A$ and $B$ are each bases of $V$.

   We first prove that $A, B$ are independent. $\alpha a_1 + \beta a_2 = \beta t^2 + \alpha t + (-\alpha - \beta)$; if this equals zero then $\alpha = \beta = 0$. $\alpha b_1 + \beta b_2 = (\alpha + \beta)t^2 + (\alpha + 2\beta)t + (-2\alpha - 3\beta)$; if this equals zero then $\alpha + \beta = 0 = \alpha + 2\beta$. The only solution is $\alpha = \beta = 0$.

   Because $V \neq \mathbb{R}_2[t]$, which has dimension 3, $V$ has dimension at most 2. However, it has dimension at least 2 since $A, B$ are in $V$, independent, and of cardinality 2. Hence $A, B$ are bases.

2. Calculate $[3t^2 - 5t + 2]_A$.

   Since only $a_2$ has a $t^2$ term, and only $a_1$ has a $t$ term, this is easy: $(-5, 3)^T$.

   Note: this means that $3t^2 - 5t + 2 = (-5)(t - 1) + (3)(t^2 - 1)$.

3. Calculate $P_{BA}$.

   It is easier to first find $P_{AB} = (b_1|_A b_2|_A) = ((1, 1)^T (2, 1)^T) = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$.

   We now calculate $P_{BA} = P_{AB}^{-1} = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$.

4. Use the results of the previous two problems to calculate $[3t^2 - 5t + 2]_B$.

   $[3t^2 - 5t + 2]_B = P_{BA}[3t^2 - 5t + 2]_A = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} 11 \\ -8 \end{pmatrix}$.

   Note: this means that $3t^2 - 5t + 2 = (11)(t^2 + t - 2) + (-8)(t^2 + 2t - 3)$.

5. Let $W = \{ at : a \in \mathbb{R} \}$. This is a subspace of $\mathbb{R}_2[t]$. Prove that $\mathbb{R}_2[t]$ is the internal direct sum of $V$ and $W$.

   This is a consequence of Theorem 2.13. We have been told (and we believe, being trusting people) that $V, W$ are subspaces of $\mathbb{R}_2[t]$. To complete the proof, we need to show two things:

   1. The dimension of $V$ (already calculated to be 2), plus the (unknown) dimension of $W$, equals the dimension of $\mathbb{R}_2[t]$ (already known to be 3).
   2. $V \cap W = \{ 0 \}$.

   We prove that $W$ is one-dimensional (1) by observing that every polynomial in $W$ is a scalar multiple of every other; hence an independent set can have only one vector in it. We next note that for $f(t) = at$, an element of $W$, $f(1) = a$. Hence for this to be in $V$ we must have $a = 0$; in this case $f(t) = 0$ which is the zero polynomial (zero vector). This proves (2).