Math 524 Exam 10 Solutions

1. Find all $2 \times 2$ complex matrices that are simultaneously diagonal, Hermitian, and unitary.

   Let $A = \left(\begin{array}{cc} a & 0 \\ 0 & b \end{array}\right)$, a diagonal matrix with complex entries. Its eigenvalues are precisely $a, b$. Because $A$ is Hermitian, they must be real. Because $A$ is unitary, they must each be of absolute value 1. There are exactly four matrices satisfying these conditions: $\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array}\right), \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right)$. They satisfy the requirements, since for each of them $A^* = A = A^{-1}$.

2. Find all $2 \times 2$ real matrices that are simultaneously symmetric and orthogonal.

   Let $A = \left(\begin{array}{cc} a & b \\ b & c \end{array}\right)$, a symmetric matrix with real entries. Because $A$ is symmetric and orthogonal (much as in problem 1), the eigenvalues must each be 1 or $-1$. Hence $|A| = \pm 1$. Consider first $|A| = 1$. Then $\frac{1}{\sqrt{1}} \left(\begin{array}{cc} c & -b \\ -b & a \end{array}\right) = A^{-1} = A^T = A = \left(\begin{array}{cc} a & b \\ b & c \end{array}\right)$. Hence $a = c, b = -b$; the only such matrices are $\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array}\right)$. These two both satisfy the requirements, as in problem 1. Now consider $|A| = -1$. Then $\frac{1}{\sqrt{-1}} \left(\begin{array}{cc} c & -b \\ -b & a \end{array}\right) = A^{-1} = A^T = A = \left(\begin{array}{cc} a & b \\ b & c \end{array}\right)$. Hence $a = -c, -1 = |A| = -a^2 - b^2$, so we can write $A = \left(\begin{array}{cc} \sin \theta & \cos \theta \\ -\cos \theta & -\sin \theta \end{array}\right)$. This satisfies the requirements for every angle $\theta$: it is obviously symmetric, and the columns (taken as vectors) are orthonormal hence the matrix is orthogonal.

   Note: this includes the diagonal matrices from problem 1, for $\theta = \pm \pi/2$.

3. Find all $2 \times 2$ complex matrices that are simultaneously anti-symmetric and unitary.

   Let $A = \left(\begin{array}{cc} 0 & a \\ -a & 0 \end{array}\right)$, an anti-symmetric matrix with complex entries. (note: $\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) = -\left(\begin{array}{cc} a & c \\ b & d \end{array}\right)$ yields $a = d = 0, b = -c$). Since $A$ is unitary, its eigenvalues must each be of unit length, hence $|A| = a^2$ must be of unit length, so $|a| = 1$. We write $a = e^{i\theta}$; so $A = \left(\begin{array}{cc} 0 & e^{i\theta} \\ -e^{i\theta} & 0 \end{array}\right)$. For any $\theta$, this satisfies the requirements: it is anti-symmetric by inspection, and unitary since the columns are orthonormal.

4. Find the maximum of $2x^2 + 4xy + 5y^2$ subject to $x^2 + y^2 = 1$.

   The quadratic function $f_A(x, y) = 2x^2 + 4xy + 5y^2$ is $\langle x | A | y \rangle$, for $A = \left(\begin{array}{cc} 2 & 2 \\ 2 & 5 \end{array}\right)$. This has eigenvalues 1, 6. $\lambda = 1$ has eigenvector $\left(\begin{array}{c} 1 \\ 1 \end{array}\right)$, which we divide by its length to satisfy the $x^2 + y^2 = 1$ condition: $\frac{1}{\sqrt{2}} \left(\begin{array}{c} 1 \\ 1 \end{array}\right)$. $\lambda = 6$ has eigenvector $\left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right)$, which we divide by its length to satisfy the $x^2 + y^2 = 1$ condition: $\frac{1}{\sqrt{2}} \left(\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array}\right)$. By the methods show in the book, one of these must maximize our quadratic function. $f_A(2/\sqrt{3}, -1/\sqrt{3}) = 1, f_A(1/\sqrt{3}, 2/\sqrt{3}) = 6$. Hence the maximum is 6.