1. State the eight axioms of a vector space.

Vector addition is an abelian group. That is, addition is commutative, associative, with an identity and inverses. More precisely:
\[ x + y = y + x, \quad x + (y + z) = (x + y) + z \]
for all vectors \( x, y, z \); there is a vector \( 0 \) such that \( 0 + x = x \) for all \( x \); every vector \( x \) has an associated \( -x \) such that \( x + (-x) = 0 \).

Scalar multiplication respects the associated field. More precisely:
For every vector \( x \) and scalars \( a, b \),
\[ 1(x) = x, \quad (ab)x = a(bx). \]
Finally, two distributive laws hold. For every scalars \( a, b \) and vectors \( x, y \),
\[ a(x + y) = ax + ay, \quad (a + b)x = ax + bx. \]

2. Using only the eight vector space axioms, prove that an element of a vector space has at most one additive inverse.

Suppose that for some vector \( x \) there were two inverses \( y, z \). Then \( x + y = 0 = x + z \).
We have \( z = 0 + z = (x + y) + z = y + (x + z) = y + 0 = 0 + y = y \), where we used (in order) the identity axiom, the hypothesis, commutativity, associativity, the hypothesis, commutativity, and the identity axiom again.

3. In \( \mathbb{R}^2 \), is it possible to have a set of two vectors that is:
   
   (a) independent and spanning
   (b) not independent and spanning
   (c) independent and not spanning
   (d) not independent and not spanning

Solution 1 (theorem): Writing the two vectors as columns of a \( 2 \times 2 \) matrix, we apply Theorem 2.5 from the text; they are independent if and only if they span \( \mathbb{R}^2 \). Hence (a) and (d) are possible, while (b) and (c) are not.

Solution 2 (dimension): If two vectors were independent, the subspace they span would be of dimension 2 hence would span \( \mathbb{R}^2 \). If two vectors were not independent, the subspace they span would be of dimension at most 1 and hence not all of \( \mathbb{R}^2 \).

4. Find the general solution to the coupled system of differential equations given by \( \frac{d^2}{dt^2} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 & 5 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \). Hint: try \( y_1 = x_1 + x_2, y_2 = x_1 - x_2 \).

You may leave the constants as constants, you need not find them in terms of \( x(0) \).

The suggested substitution leads to \( \frac{d^2}{dt^2} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -6 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \), which has solutions \( y_1(t) = c_1 e^{2t} + c_2 e^{-2t}, y_2(t) = c_3 \cos(\sqrt{6}t) + c_4 \sin(\sqrt{6}t) \). We have \( x_1 = (y_1 + y_2)/2, x_2 = (y_1 - y_2)/2 \), so (setting \( d_i = c_i/2 \)) we get
\[ x_1(t) = d_1 e^{2t} + d_2 e^{-2t} + d_3 \cos(\sqrt{6}t) + d_4 \sin(\sqrt{6}t), \]
\[ x_2(t) = d_1 e^{2t} + d_2 e^{-2t} - d_3 \cos(\sqrt{6}t) - d_4 \sin(\sqrt{6}t). \]