Math 522 Exam 10 Solutions

1. Prove that \( \sum_{r|n} \frac{\mu(r)}{d(r)} = 2^{-s} \), where \( s \) is the number of different primes dividing \( n \), i.e. \( n = p_1^{a_1}p_2^{a_2} \cdots p_s^{a_s} \).

Set \( G(n) = \sum_{r|n} \frac{\mu(r)}{d(r)} \). Note that \( \frac{\mu}{d} \) is multiplicative, since both the numerator and denominator are (and the denominator is never zero). Because \( G = \frac{\mu}{d} \ast 1 \), \( G \) is also multiplicative. Hence \( G(n) = \prod_{i=1}^{s} G(p_i^{a_i}) = \prod_{i=1}^{s} \left( \frac{\mu(1)}{d(1)} + \frac{\mu(p_i)}{d(p_i)} + \frac{\mu(p_i^2)}{d(p_i^2)} + \cdots \right) = \prod_{i=1}^{s} \left( 1 + \frac{-1}{2} + 0 + \cdots \right) = \prod_{i=1}^{s} \left( \frac{1}{2} \right) = 2^{-s} \).

The number of different primes dividing \( n \) is called \( \omega(n) \), which is interesting in its own right. This is an additive function (not multiplicative); however exponentiating an additive function makes a multiplicative function.

2. Prove that \( d^{-1} = \mu \ast \mu \). Compute \( d^{-1}(27) \), either using this fact or recursively.

We begin with \( 1 \ast 1 = d \), and multiply both sides by \( d^{-1} \ast \mu \ast \mu \) to get \( d^{-1} \ast \mu \ast \mu \ast 1 \ast 1 = d \ast d^{-1} \ast \mu \ast \mu \). This simplifies to \( d^{-1} = \mu \ast \mu \), because \( d \ast d^{-1} = I = 1 \ast \mu \) and \( \ast \) is commutative and associative.

Using this fact, \( (\mu \ast \mu)(27) = \sum_{d \leq 27} \mu(d) \mu(d) = \mu(1)\mu(27) + \mu(3)\mu(9) + \mu(9)\mu(3) + \mu(27)\mu(1) = 0 \), since \( \mu(27) = \mu(9) = 0 \).

Recursively, we calculate \( d^{-1}(3) \), \( d^{-1}(9) \), \( d^{-1}(27) \).
\[
\begin{align*}
d^{-1}(3) &= -\sum_{r|3, r<3} d(r)^{-1}(r) = -d(3)d^{-1}(1) = -2.
d^{-1}(9) &= -(d(9)d^{-1}(1) + d(3)d^{-1}(3)) = -(3 - 4) = 1.
d^{-1}(27) &= -(d(27)d^{-1}(1) + d(9)d^{-1}(3) + d(3)d^{-1}(9)) = -(4 - 6 + 2) = 0.
\end{align*}
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