1. For \( f(x) = 47x^2 + x - 2 \), find all solutions to \( f(x) \equiv 0 \pmod{47^2} \).

BONUS: Find all solutions to \( f(x) \equiv 0 \pmod{47^3} \).

We use the lifting theorem, so we first solve \( f(x) \equiv 0 \pmod{47} \). Conveniently, \( 47x^2 + x - 2 \equiv x - 2 \pmod{47} \), so \( x = 2 \) is the unique root modulo \( 47 \). Now, \( f'(x) = 94x + 1 \), so \( f'(2) = 189 \equiv 1 \pmod{47} \), so this root will lift to a unique root modulo \( 47^2 \). We solve \( f'(2)t \equiv -f(2)/47 \pmod{47} \), which simplifies to \( 1t \equiv -188/47 = -4 \equiv 43 \pmod{47} \). Hence \( x = 2 + 43 \cdot 47 = 2023 \) is the unique root of \( f(x) \) modulo \( 47^2 = 2209 \).

BONUS: We start with the sole root \( r = 2023 \), modulo \( 47^2 \). We have \( f'(2023) = 94 \cdot 2023 + 1 \equiv 1 \pmod{47} \), so again this root will lift uniquely modulo \( 47^3 \). We solve \( f'(2023)t \equiv -f(2023)/47^2 \pmod{47} \), which simplifies to \( 1t \equiv -87076 = 15 \pmod{47} \). Hence \( x = 2023 + 15 \cdot 47^2 = 35158 \) is the unique root of \( f(x) \) modulo \( 47^3 = 103823 \).

2. For \( n \in \mathbb{N} \), prove that \( \phi(n) \) is even if and only if \( n > 2 \).

Suppose that \( p^a \mid n \) for any odd prime \( p \) and \( a \in \mathbb{N} \), then (since \( \phi \) is multiplicative) we take \( a \) maximal and have \( \phi(n) = \phi(p^a) \phi\left(\frac{n}{p^a}\right) = (p^a - p^{a-1}) \phi\left(\frac{n}{p^a}\right) \).

But \( p^a \) is odd, and so is \( p^{a-1} \), so their difference is even, and so hence is \( \phi(n) \). Hence \( \phi(n) \) is even for every \( n \) that is not a power of 2 (powers of 2 have not yet been addressed). Now \( \phi(2^a) = 2^a - 2^{a-1} \). This is even for \( a \geq 2 \), being the difference of two even numbers. Hence \( \phi(n) \) is even for every \( n \) except possibly \( n = 1, 2 \). But in fact \( \phi(1) = \phi(2) = 1 \), which are odd.

3. High score=101, Median score=77, Low score=50