Math 522 Exam 1 Solutions

1. Please calculate the number corresponding to the representation (3, 2, 1), i.e. 321, in ten ways: in base 4, 8, 9, 10, 11, 12, 13, 14, 16, 35.
   BONUS: repeat for the factoradic basis.
   \[(321)_4 = 3 \cdot 4^2 + 2 \cdot 4 + 1 = (57)_{10} \text{ fifty-seven}\]
   \[(321)_8 = 3 \cdot 8^2 + 2 \cdot 8 + 1 = (209)_{10} \text{ two hundred nine}\]
   \[(321)_9 = 3 \cdot 9^2 + 2 \cdot 9 + 1 = (262)_{10} \text{ two hundred sixty-two}\]
   \[(321)_{10} = 3 \cdot 10^2 + 2 \cdot 10 + 1 = (321)_{10} \text{ three hundred twenty-one}\]
   \[(321)_{11} = 3 \cdot 11^2 + 2 \cdot 11 + 1 = (386)_{10} \text{ three hundred eighty-six}\]
   \[(321)_{12} = 3 \cdot 12^2 + 2 \cdot 12 + 1 = (457)_{10} \text{ four hundred fifty-seven}\]
   \[(321)_{13} = 3 \cdot 13^2 + 2 \cdot 13 + 1 = (534)_{10} \text{ five hundred thirty-four}\]
   \[(321)_{14} = 3 \cdot 14^2 + 2 \cdot 14 + 1 = (617)_{10} \text{ six hundred seventeen}\]
   \[(321)_{16} = 3 \cdot 16^2 + 2 \cdot 16 + 1 = (801)_{10} \text{ eight hundred one}\]
   \[(321)_{35} = 3 \cdot 35^2 + 2 \cdot 35 + 1 = (3746)_{10} \text{ thirty-seven hundred forty-six}\]
   \[(321)_F = 3 \cdot 6 + 2 \cdot 2 + 1 = (23)_{10} \text{ twenty-three}\]

2. Let \((a_s, \ldots, a_1, a_0)\) and \((a'_s, \ldots, a'_1, a'_0)\) be two \((s + 1)\)-digit representations in that basis, corresponding to numbers \(A\) and \(A'\), respectively. Suppose that \(a_i \geq a'_i\) for all \(i \in [0, s]\). Prove that \(A \geq A'\).

   METHOD 1: We prove this by induction on \(s\). For \(s = 0\), \(A = a_0 \geq a'_0 = A'\), as desired. For \(s > 0\), we split the numbers as follows: \(A = C + D, A' = C' + D'\), where \(C = a_s(b_s \cdots b_1), D = a_{s-1}(b_{s-1} \cdots b_1) + \cdots + a_1 b_1 + a_0, C' = a'_s(b_s \cdots b_1), D' = a'_{s-1}(b_{s-1} \cdots b_1) + \cdots + a'_1 b_1 + a'_0\). Now, \(C/C' = (a_s/a'_s) \geq 1\) [note \(a'_s \neq 0\) because it’s the leading digit], so \(C \geq C'\). Now, \(D\) and \(D'\) have representations \((a_{s-1}, \ldots, a_0)\) and \((a'_{s-1}, \ldots, a'_0)\); hence, by the inductive hypothesis \(D \geq D'\). Adding, we get \(A = C + D \geq C' + D' = A'\), as desired.

   METHOD 2: We have \(A - A' = (a_s(b_s \cdots b_1) + \cdots a_1 b_1 + a_0) - (a'_s(b_s \cdots b_1) + \cdots a'_1 b_1 + a'_0) = (a_s - a'_s)(b_s \cdots b_1) + \cdots (a_1 - a'_1)b_1 + (a_0 - a'_0) \geq 0\) since each summand is nonnegative, hence \(A \geq A'\).

3. High score=103, Median score=75, Low (nonblank) score=55