Math 522 Exam 7 Solutions

1. Find a maximal set of incongruent solutions to $42x \equiv 12 \pmod{450}$.

   We factor $42 = 6\cdot 7, 12 = 6\cdot 2, 450 = 6\cdot 75$. *We apply Thm 5-1 with* $d = \gcd(42, 450) = 6$ *to conclude that there should be 6 incongruent solutions. To find them, we observe that the congruence is equivalent to* $7x \equiv 2 \pmod{75}$. *At this point, we use trial and error – we multiply small integers by 7, reduce modulo 75, and hope to get either 1 (in which case we found a reciprocal of 7), or 2 (in which case we found a solution). Fortunately, we don't have to go too far, as* $x = 11$ *is a solution. Hence,* $\{11 + 75k : k \in \mathbb{Z}\}$ *is the complete set of integer solutions, and* $\{11, 86, 161, 236, 311, 386\}$ *is a maximal set of incongruent solutions.*

2. Prove that there exist three consecutive integers, each divisible by a perfect square greater than one.
   BONUS: Find such integers.
   DOUBLE BONUS: Find the smallest positive such integers.

   The strategy is to use the Chinese Remainder Theorem. Set $m_1 = 4, m_2 = 9, m_3 = 25$ *(note: many other choices are possible). We seek solutions to* $x \equiv 0 \pmod{4}, x \equiv -1 \pmod{9}, x \equiv -2 \pmod{25}$. *Any simultaneous solution to these three equivalences will have* $4|x, 9|(x+1), 25|(x+2)$. *But we are guaranteed a solution, since* $4, 9, 25$ *are pairwise relatively prime.*

   BONUS: Using the above system, we first need to solve $b_1 225 \equiv 1 \pmod{4}, b_2 100 \equiv 1 \pmod{9}, b_3 36 \equiv 1 \pmod{25}$. *We simplify these to be* $b_1 \cdot 1 \equiv 1 \pmod{4}, b_2 \cdot 1 \equiv 1 \pmod{9}, b_3 \cdot 11 \equiv 1 \pmod{25}$. *Trial and error finds* $b_3 = 16 (\equiv -9)$ *works. We put this all together to find* $x = b_1({}^m/_{m_1})a_1 + b_2({}^m/_{m_2})a_2 + b_3({}^m/_{m_3})a_3 = 1 \cdot 225 \cdot 0 + 1 \cdot 100(-1) + (-9)(36)(-2) = 548$. *To double-check,* $4|548, 9|549, 25|550$, *so* $\{548, 549, 550\}$ *are a solution.*

   DOUBLE BONUS: The simplest method is to list all the natural numbers that are not multiples of any square, and look for a gap of size at least 3. 1,2,3,5,6,7,10,11,13,14,15, 17,19,21,22,23,26,29,30,31,33,34,35,37,38,39,41,42,43,46,47,50,51,53,....
   Hence $\{48, 49, 50\}$ *is the desired set, where* $4|48, 49|49, 25|50$.

   This can also be done with a sieve (much like the sieve of Eratosthenes to find primes), where you list the numbers and then cross out all the multiples of $2^2, 3^2, 5^2, 7^2, 11^2$ *and so on. It takes a bit of a guess about how far to go – a conservative approach would go up to 900 (*$2^2 3^2 5^2$*), a risky but faster approach would go up to 100 (in which case you only need to sieve out 4,9,25, and 49).*

3. Exam grades: 106, 105, 105, 100, 100, 99, 95, 93, 87, 77, 65, 62, 61