1. Find a maximal set of incongruent solutions to $42x \equiv 12 \pmod{450}$.

We factor $42 = 6 \cdot 7$, $12 = 6 \cdot 2$, $450 = 6 \cdot 75$. We apply Thm 5-1 with $d = \gcd(42, 450) = 6$ to conclude that there should be 6 incongruent solutions. To find them, we observe that the congruence is equivalent to $7x \equiv 2 \pmod{75}$. At this point, we use trial and error – we multiply small integers by 7, reduce modulo 75, and hope to get either 1 (in which case we found a reciprocal of 7), or 2 (in which case we found a solution). Fortunately, we don't have to go too far, as $x = 11$ is a solution. Hence, $\{11 + 75k : k \in \mathbb{Z}\}$ is the complete set of integer solutions, and $\{11, 86, 161, 236, 311, 386\}$ is a maximal set of incongruent solutions.

2. Prove that there exist three consecutive integers, each divisible by a perfect square greater than one.

BONUS: Find such integers.

DOUBLE BONUS: Find the smallest positive such integers.

The strategy is to use the Chinese Remainder Theorem. Set $m_1 = 4, m_2 = 9, m_3 = 25$ (note: many other choices are possible). We seek solutions to $x \equiv 0 \pmod{4}, x \equiv -1 \pmod{9}, x \equiv -2 \pmod{25}$. Any simultaneous solution to these three equivalences will have $4|\ x, \ 9| (x+1), \ 25|(x+2)$. But we are guaranteed a solution, since 4, 9, 25 are pairwise relatively prime.

BONUS: Using the above system, we first need to solve $b_1225 \equiv 1 \pmod{4}, b_2100 \equiv 1 \pmod{9}, b_336 \equiv 1 \pmod{25}$. We simplify these to be $b_1 \cdot 1 \equiv 1 \pmod{4}, b_2 \cdot 1 \equiv 1 \pmod{9}, b_3 \cdot 11 \equiv 1 \pmod{25}$. Trial and error finds $b_3 = 16(= -9)$ works. We put this all together to find $x = b_1(m/m_1)a_1 + b_2(m/m_2)a_2 + b_3(m/m_3)a_3 = 1 \cdot 225 \cdot 0 + 1 \cdot 100(-1) + (-9)(36)(-2) = 548$. To double-check, $4|548, 9|549, 25|550$, so $\{548, 549, 550\}$ are a solution.

DOUBLE BONUS: The simplest method is to list all the natural numbers that are not multiples of any square, and look for a gap of size at least 3. 1, 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23, 26, 29, 30, 31, 33, 34, 35, 37, 38, 39, 41, 42, 43, 46, 47, 50, 51, 53, … Hence $\{48, 49, 50\}$ is the desired set, where $4|48, 49|49, 25|50$.

This can also be done with a sieve (much like the sieve of Eratosthenes to find primes), where you list the numbers and then cross out all the multiples of $2^2, 3^2, 5^2, 7^2, 11^2$ and so on. It takes a bit of a guess about how far to go – a conservative approach would go up to 900 ($2^23^25^2$), a risky but faster approach would go up to 100 (in which case you only need to sieve out 4, 9, 25, and 49).

3. Exam grades: 106, 105, 105, 100, 100, 99, 95, 93, 87, 77, 65, 62, 61