Math 522 Exam 5 Solutions

1. Prove or disprove: \(29^{76} \equiv 76^{29} \pmod{35}\).

   Easy way: Statement holds if and only if \(29^{76} \equiv 76^{29} \pmod{5}\) AND \(29^{76} \equiv 76^{29} \pmod{7}\). Note that, modulo 5, \(29 \equiv -1\) and \(76 \equiv 1\). Also, modulo 7, \(29 \equiv 1\) and \(76 \equiv -1\). The first equation therefore simplifies to \((-1)^{76} \equiv (1)^{29} \pmod{5}\), which is true since \((-1)^{76} = 1\). The second equation, however, simplifies to \((1)^{76} \equiv (-1)^{29} \pmod{7}\), which is false. Hence the statement does not hold.

   Note: It was not necessary to check modulo 5, I did this for completeness.

   Hard way: Working modulo 35, we see that \(29 \equiv -6\) and \(76 \equiv 6\). Further, \(29^{76} \equiv (-6)^{76} \equiv ((-6)^2)^{38} \equiv 6^{76}\). On the other hand, \(76^{29} \equiv 6^{29}\). Hence, the problem is equivalent to \(6^{76} \equiv 6^{29} \pmod{35}\). Let’s calculate powers of 6, modulo 35. Immediately we see that \(6^2 \equiv 1\). Hence \(6^{28} \equiv (6^2)^{14} \equiv 1\). But then \(6^{29} = 6^{28} \cdot 6 = \not{1} = (6^2)^{38} = 6^{76}\).

   Completely mechanical way: We first calculate powers of 29, modulo 35. We immediately see that \(29^2 \equiv 1\), and hence \(29^{76} = (29^2)^{37} \equiv 1\). We now calculate powers of 76, modulo 35. We again find the extremely lucky situation that \(76^2 \equiv 1\), and hence \(76^{29} \equiv (76^2)^{14} \cdot 76^1 \equiv 76\). It remains to check whether \(1 \equiv 76 \pmod{35}\), which is false.

2. Recall that \(a \equiv b \pmod{c}\) means that \((a - b)/c\) is an integer. Recall also that \(\lfloor \alpha \rfloor\) is the greatest integer less than or equal to \(\alpha\). For real numbers \(x, y\) \((y > 0)\) define \(f(x, y) = x - y\lfloor x/y \rfloor\).

   (a) Prove that \(f(x, y) \equiv x \pmod{y}\).

   We calculate \((f(x, y) - x)/y = (x - y\lfloor x/y \rfloor - x)/y = -y\lfloor x/y \rfloor/y = -\lfloor x/y \rfloor\). But this is an integer, by the definition of \(\lfloor \alpha \rfloor\).

   (b) Prove that \(0 \leq f(x, y) < y\)

   The key fact is that \(\alpha - 1 < \lfloor \alpha \rfloor \leq \alpha\), for any real number \(\alpha\). Because \(\lfloor \alpha \rfloor \leq \alpha\), we have \(-\lfloor \alpha \rfloor \geq -\alpha\). Hence \(f(x, y) = x - y\lfloor x/y \rfloor \geq x - y\lfloor x/y \rfloor = x - x = 0\). On the other hand, \(y - f(x, y) = y - x + y\lfloor x/y \rfloor > y - x + y(\lfloor x/y \rfloor - 1) = y - x + x - y = 0\).

   Alternate solution: Use the similar fact, that \(\alpha - \lfloor \alpha \rfloor = \beta\), for some \(0 \leq \beta < 1\). Hence \(\lfloor x/y \rfloor = x/y - \beta\). We substitute this into \(f(x, y) = x - y(x/y - \beta) = y\beta\). Since \(0 \leq \beta < 1\) and \(y > 0\), the desired results follow.

3. Exam grades: 100, 95, 94, 93, 87, 84, 81, 76, 72, 68, 65, 63, 52