We now define a very important infinite non-abelian group, with operation matrix multiplication. The general linear group is $GL(2, \mathbb{R}) = \{(a \ b \ c \ d) : a, b, c, d \in \mathbb{R}, ad - bc \neq 0\}$.

1. Set $H = \{(a \ b \ c \ d) : a, b, c, d \in \mathbb{R}, ad - bc > 0\}$. Prove that $H \trianglelefteq GL(2, \mathbb{R})$.

2. Set $K = \{(a \ b \ c \ d) : a, b, c, d \in \mathbb{R}, ad - bc \in \mathbb{Q}, ad - bc \neq 0\}$. Prove that $K \trianglelefteq GL(2, \mathbb{R})$.

3. Set $M = \{(a \ b \ 0 \ d) : a, b, d \in \mathbb{R}, ad \neq 0\}$. Prove that $M \not\trianglelefteq GL(2, \mathbb{R})$.

4. Set $S = \{(a \ b \ c \ d) : a, b, c, d \in \mathbb{R}, ad - bc = 1\}$. Prove that $S \trianglelefteq GL(2, \mathbb{R})$. This is called the special linear group and is denoted $SL(2, \mathbb{R})$.

5. Find the center $Z$ of $SL(2, \mathbb{R})$. The quotient group $SL(2, \mathbb{R})/Z$ is called the projective special linear group and is denoted $PSL(2, \mathbb{R})$. It turns out to be isomorphic to the group of all complex Möbius transformations (aka linear fractional transformations) i.e. $f(z) = \frac{az+b}{cz+d}$.

We now turn to general groups $G$.

6. Suppose that $H \leq G$, and let $a \in G$. Prove that $aHa^{-1} \leq G$, and $|H| = |aHa^{-1}|$.

7. Suppose that $H \leq G$. Suppose that $H$ is the only subgroup of $G$ of order $|H|$. Prove that $H \trianglelefteq G$.

8. Suppose that $N \leq G$. Prove that $N \trianglelefteq G$ if and only if the product of any two right cosets of $N$ is another right coset of $N$.

9. Suppose that $N \leq G$. Prove that $N \trianglelefteq G$ if and only if every left coset of $N$ is a right coset of $N$.

10. Prove that $A_n \trianglelefteq S_n$, where $S_n$ is the symmetric group and $A_n$ is the alternating group (set of even permutations).

We recall the quaternion group $Q$, as defined in the previous homework.

11. Set $K = \{1, -1, i, -i\}$. Write down the multiplication table for the quotient group $Q/K$.

12. Set $H = \{1, -1\}$. Write down the multiplication table for the quotient group $Q/H$.

13. A group all of whose subgroups are normal is called a Dedekind group. Prove that $Q$ is a non-abelian Dedekind group.