MATH 521B: Abstract Algebra
Homework 5: Due Feb. 23

Let S be a finite set, and let G be a permutation group on S. For every $s \in S$, we define its *orbit* under G as $Gs = \{gs : g \in G\}$. In this way G induces a partition on S (see #2). The number of orbits, i.e. the number of parts in the partition, we denote with $|S/G|$. For every $g \in G$, we define its *fixture* as $S^g = \{s \in S : gs = s\}$, i.e. those elements of S that are fixed under g. The following is popularly known as Burnside's Lemma, although it was proved by Cauchy first, and even Burnside himself credited it to Frobenius.

$$|S/G| = \frac{1}{|G|} \sum_{g \in G} |S^g|$$

The most common use of Burnside's Lemma is in coloring problems. We have an object and want to color its parts. However we want to know how many "different" colorings there are, up to various allowed permutations of the object. S consists of all the valid colorings of the object, and G is the set of permutations. Each orbit is considered the "same" coloring, so we want to count the number of orbits.

1. Suppose that $a \in Gb$. Prove that $Ga = Gb$.
2. Prove that the set of orbits partition the set S, i.e. are disjoint sets whose union is S.
3. Prove that $S^g \cap S^h \subseteq S^{(gh)}$, for all $g, h \in G$. Find an example where the containment is strict.
4. Use Burnside's Lemma to calculate how many different ways there are to color the vertices of a non-square rectangle black or white, up to all isometries.
5. Use Burnside's Lemma to calculate how many different ways there are to color the vertices of a square black or white, up to all isometries.
6. Use Burnside's Lemma to calculate how many different ways there are to color the vertices of a square black, white, or green, up to all isometries.
7. Use Burnside's Lemma to calculate how many different ways there are to color the vertices of a square black, white, or green, up to rotation only.
8. Use Burnside's Lemma to calculate how many different ways there are to color the vertices of a regular pentagon black, white, or green, up to all isometries. (hint: $|G| = 10$).
9. Use Burnside's Lemma to calculate how many different ways there are to color the vertices of a regular pentagon black, white, or green, up to rotation only.
10. Use Burnside's Lemma to calculate how many different ways there are to color the faces of a physical cube black or white, up to physically possible isometries. (you can't turn a physical cube inside out, $|G| = 24$)
11. Use Burnside's Lemma to calculate how many different ways there are to color the corners of a physical cube black or white, up to physically possible isometries.
12. Use Burnside's Lemma to calculate how many different ways there are to color the edges of a physical cube black or white, up to physically possible isometries.

Note: Problems 4,5,10 have small answers and it may be fun to draw them all. Also, to draw the solutions to #7 that are NOT solutions to #6.