**MATH 521B: Abstract Algebra**
Homework 11: Due Apr. 27

Since this is the last homework, I will make the folowing special offer: if you get 10/10 on this homework, I will count it double in computing your homework grade.

1. If $G$ is a finite abelian group, and prime $p$ divides $|G|$, prove that $G$ has an element of order $p$.

2. If $G$ is a finite abelian group, and $p^n$ divides $|G|$ (for prime $p$), prove that $G$ has a subgroup of order $p^n$.

3. Determine which positive integers $n$ have exactly one abelian group of order $n$ (up to isomorphism).

   For problems 4-6, consider the finitely generated abelian groups given. Find the Smith Normal Form of the matrices of relations, and use this to find groups in standard form, isomorphic to the given groups.

4. $G = \langle a, b, c, d : 3a + 3b - 3c + 6d = 0, -3a + 3c - 6d = 0, -3a + 3b + 3c - 6d = 0, 9a + 9b - 9c + 18d = 0\rangle$.

5. $G = \langle a, b, c, d : 3a + 3b - 3c + 6d = 0, -3a + 6c - 9d = 0, 3a + 9b + 9c + 6d = 0, 9a + 21b - 3c = 0\rangle$.

6. $G = \langle a, b, c, d : 3a+3b-3c+6d = 0, -3a+6c = 0, 3a+6b+12d = 0, 7a+9b-3c+18d = 0\rangle$.

7. Prove that if $G$ is a finite abelian group with $|G| \leq 2$, then every element in $\mathcal{B}(G)$ has exactly one factorization into irreducibles.

8. Prove that if $G$ is a finite abelian group with $|G| > 2$, then there is some element in $\mathcal{B}(G)$ with at least two factorizations into irreducibles.

9. Find an element $x \in \mathcal{B}(\mathbb{Z}_n)$ such that $x$ has a factorization into both $n$ irreducibles, and also 2 irreducibles. This ratio $(\frac{n}{2})$ is called the *elasticity* of $x$.

10. Prove that $\mathcal{B}(\mathbb{Z}_n)$, the block monoid on group $\mathbb{Z}_n$, has at least $p(n)$ irreducibles, where $p(n)$ denotes the number of integer partitions of $n$.

11. Prove that $D(\mathbb{Z}_n) = n$, i.e. $D(G) = D^\star(G)$ for rank-1 groups $G$.