MATH 521B: Abstract Algebra Final Exam

Please read the following instructions. For the following exam you may use a single notesheet, but no books, papers, calculator, or computers. Please turn in **exactly ten** problems. You must do problems 1-6, and four more chosen from 7-12. Please write your answers on separate paper, make clear what work goes with which problem, and put your name or initials on every page. You have 120 minutes. Each problem will be graded on a 5-10 scale (as your quizzes), for a total score between 50 and 100. Bonus points are in addition; if you do problems 9-12 and also all four bonus problems your maximum possible score is 108.

Turn in problems 1,2,3,4,5,6:

1. Find a subgroup $H \leq S_5$, such that $|H| = 6$.
2. Let G be a group. We recall the center $Z(G) = \{x \in G : \forall y \in G, xy = yx\}$. Prove that $Z(G) \leq G$.
3. Carefully state and prove Lagrange's Theorem (the one from this course).
4. Let G be a group, with $K \leq G$ and $a \in G$. Prove that $|K| = |Ka|$.
5. Consider the abelian group G, written additively, whose presentation follows. Determine the betti number and invariant factors, and find an isomorphic group in standard form (a direct sum of cyclic groups). $G = \langle a, b | a = 2b, 3a = 0 = 6b\rangle$
6. Let $p, m, n \in \mathbb{N}$, with p prime and $m < p$. Prove that there is no simple group of order $p^n m$.

Turn in exactly four more problems of your choice:

7. Suppose G is a group with the property that $(ab)^2 = a^2 b^2$ for all $a, b \in G$. Prove that G is abelian.
8. Find the elementary divisors and invariant factors of $\mathbb{Z}_{5040}^{\times}$. Hint: $5040 = 7!$.
9. Consider the group of isometries of an equilateral triangle. Give all of its subgroups. (2 bonus points if you also determine which are normal)
10. We have three colors at our disposal, and color the edges of an equilateral triangle. Use Burnside's lemma to count how many ways there are to do this, different up to isometries of the triangle. (2 bonus points if you repeat the solution, with n colors)
11. Give all nonisomorphic abelian groups of order 225. For each, give the elementary divisors and invariant factors. (2 bonus points if you also find the Davenport constant of each)
12. Carefully state and prove the First Isomorphism Theorem. You may freely use the following result, without proving it: (2 bonus points if you *also* prove the lemma)
Lemma: Let G, H be groups, $f : G \to H$ a group homomorphism, and $K = ker(f)$. For all $a, b \in G$, we have $f(a) = f(b)$ if and only if $Ka = Kb$.