MATH 521B: Abstract Algebra

Exam 1 Solutions

1. Fix a group G. Let $a, b \in G$ with $ab = ba$, $|a|, |b|$ finite, and $\langle a \rangle \cap \langle b \rangle = \{id\}$. Prove that $|ab| = lcm(|a|, |b|)$.

   Set $k = lcm(|a|, |b|)$. Then $(ab)^k = a^k b^k = (a^{|a|})^{k/|a|}(b^{|b|})^{k/|b|} = (id)^{k/|a|}(id)^{k/|b|} = id$. Hence, by a lemma proved in class, $|ab|$ divides k, and in particular $|ab| \leq k$. Now, we have $id = (ab)^{|ab|} = a^{|ab|}b^{|ab|}$ so $a^{|ab|} = b^{-|ab|}$. But since $\langle a \rangle \cap \langle b \rangle = \{id\}$, we have $a^{|ab|} = b^{-|ab|} = id$. Hence $|ab|$ is a multiple of $|a|$, and is also a multiple of $|b|$, thus a multiple of k. So in particular $|ab| \geq k$. Combining, $|ab| = |k|$.

2. Prove that every group G with exactly three elements must be abelian.

   Let's name the elements as $\{id, a, b\}$. id commutes with everything, in every group. Also every element commutes with itself. So the only possibility for G to be non-abelian is if $ab \neq ba$. Now, if $ab = a$, then $a^{-1}ab = a^{-1}a$, which gives $b = id$, a contradiction. Similarly, if $ab = b$, then $abb^{-1} = bb^{-1}$, so $a = id$, a contradiction. Hence $ab = id$, and similarly $ba = id$. But now $ab = ba$, and G is abelian.

3. Let G be an abelian group. Let $f : G \to G$ be defined via $f : x \mapsto x^2$. Prove that f is a homomorphism.

   For arbitrary $x, y \in G$, we have $f(xy) = (xy)^2 = xyxy = xxyy = x^2y^2 = f(x)f(y)$.

4. Fix groups G, H, and a homomorphism $f : G \to H$. Let K be a subgroup of H. Prove that $S = \{g \in G : f(g) \in K\}$ is a subgroup of G.

   First, we show that S is closed. For arbitrary $x, y \in S$, we have $f(xy) = f(x)f(y)$. Since $f(x) \in K$ and $f(y) \in K$, we must have $f(x)f(y) \in K$. Thus $xy \in S$, and S is closed.
   Second, $S \neq \emptyset$ since $f(id) = id \in K$, so $id \in S$. This is obvious enough to be omitted, if desired.
   Lastly, we show that S contains inverses. Let $x \in S$. There is some $x^{-1} \in G$ such that $xx^{-1} = id$. We have $f(id) = f(xx^{-1}) = f(x)f(x^{-1})$, so $f(x^{-1}) = f(x)^{-1}$. Since $f(x) \in K$, and $K \leq H$, $f(x)^{-1} \in K$. Thus $f(x^{-1}) \in K$, and so $x^{-1} \in S$.

5. Fix a group G, a subgroup H, and an element $a \in G$. Define $aHa^{-1} = \{aha^{-1} : h \in H\}$. Prove that aHa^{-1} is a subgroup of G.

   First, we show that aHa^{-1} is closed. For arbitary $x, y \in aHa^{-1}$, there must be $h_x, h_y \in H$ such that $x = ah_xa^{-1}, y = ah_ya^{-1}$. We have $xy = ah_xa^{-1}ah_ya^{-1} = a(h_xh_y)a^{-1}$. Since H is a subgroup, $h_xh_y \in H$, and hence $xy \in aHa^{-1}$.
   Second, $aHa^{-1} \neq \emptyset$ since $id = a(id)a^{-1} \in aHa^{-1}$. This is obvious enough to be omitted, if desired.
   Lastly, we show that aHa^{-1} contains inverses. Let $x \in aHa^{-1}$. There must be $h_x \in H$ such that $x = ah_xa^{-1}$. Since H is a subgroup, $h_x^{-1} \in H$, so we set $y = ah_x^{-1}a^{-1}$ and calculate $xy = ah_xa^{-1}ah_x^{-1}a^{-1} = id$.

6. Fix a group G. Recall the group center $Z(G) = \{a \in G : \forall b \in G, ab = ba\}$, and the centralizer $C(x) = \{y \in G : xy = yx\}$, which is defined for each $x \in G$. Prove that $Z(G) = \bigcap_{g \in G} C(g)$.

First we show $\subseteq$. Let $x \in Z(G)$. For any $g \in G$, we have $xg = gx$ since x is in the center. Hence $x \in C(g)$ for every $g \in G$, and thus $x \in \bigcap_{g \in G} C(g)$.

Next we show $\supseteq$. Let $x \in \bigcap_{g \in G} C(g)$. In particular, for each $b \in G$, we have $x \in C(b)$, so $xb = bx$. Thus $x \in Z(G)$.

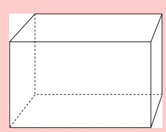
7. Fix a group G. For subgroups A, B, we define their product $AB = \{ab : a \in A, b \in B\}$. Suppose H, K are both subgroups of G that satisfy $HK = KH$. Prove that HK is a subgroup of G.

First, we show that HK is closed. For arbitrary $x, y \in HK$, we have $x = h_1k_1, y = h_2k_2$, where $h_1, h_2 \in H$ and $k_1, k_2 \in K$. We write their product $xy = h_1(k_1h_2)k_2$. Since $k_1h_2 \in KH = HK$, there must be some h_3k_3 with $k_1h_2 = h_3k_3$. Substituting, we get $xy = (h_1h_3)(k_3k_2)$. Since H, K are subgroups, $h_1h_3 \in H$ and $k_3k_2 \in K$. Thus $xy \in HK$.

Second, $HK \neq \emptyset$ since $(id)(id) = id \in HK$. This is obvious enough to be omitted, if desired.

Lastly, we show that HK contains inverses. Let $x \in HK$. We have $x = hk$, where $h \in H, k \in K$. Now, $h^{-1} \in H, k^{-1} \in K$, so $k^{-1}h^{-1} \in KH$. But since $KH = HK$, there are $h_1 \in H, k_1 \in K$ with $k^{-1}h^{-1} = h_1k_1$. Now, we set $y = h_1k_1 \in HK$, and calculate $xy = hkk^{-1}h^{-1} = id$.

8. Consider a solid square prism, i.e. shoebox, with two identical square ends and four identical rectangular (but not square) sides, as pictured below. Color each face black or white. How many different ways are there to do this, up to physically possible isometries of the solid figure?



We have $|S| = 2^6 = 64$ colorings, and $|G| = 8$. The group consists of id, rotations of $90°, 180°, 270°$ around the long axis, and four more, corresponding to swapping the left and right squares and simultaneously putting the top face in the top, front, bottom, and back positions respectively. We have $|S^{id}| = 2^6$, $|S^{90}| = 2^3 = |S^{270}|$, $|S^{180}| = 2^4$, $|S^{top}| = 2^4$, $|S^{front}| = 2^3$, $|S^{bottom}| = 2^4$, $|S^{back}| = 2^3$. Putting it all together we get $|S/G| = \frac{1}{8}(2^6 + 3 \cdot 2^4 + 4 \cdot 2^3) = \frac{144}{8} = 18$.