MATH 521B: Abstract Algebra

Preparation for Exam 1

1. Find a group consisting of complex numbers under multiplication, that is cyclic of order 5.
2. Prove that the permutation group $S_n = \langle (1\ 2\ 3\ \cdots\ n), (1\ 2) \rangle$, i.e. is generated by just those two permutations.
3. For arbitrary $n \in \mathbb{Z}$, we consider the cyclic subgroup of $(\mathbb{Z}, +)$ given by $\langle n \rangle$. For arbitrary $m, n \in \mathbb{Z}$, determine $\langle m \rangle \cap \langle n \rangle$.
4. Set $G = \langle a, b, c : a^3 = b, b^4 = c, c^5 = id \rangle$. Determine $|a|$, if it exists.
5. Fix a group G. Let $a, b \in G$ with $ab = ba$, $|a|, |b|$ finite, and $\langle a \rangle \cap \langle b \rangle = \{id\}$. Prove that $|ab| = lcm(|a|, |b|)$.
6. Prove that every group G with exactly four elements must be abelian.
7. Prove that every subgroup of a cyclic group is cyclic.
8. Fix a group G. Suppose that $|G|$ is even. Prove that there is some element $a \in G$ with $|a| = 2$.
9. Fix an abelian group G, and set $S = \{a \in G : |a| \leq 2\}$. Prove that S is a subgroup of G.
10. Fix groups G, H, and a homomorphism $f : G \to H$. Define the kernel of f as $Ker(f) = \{g \in G : f(g) = id\}$. Prove that $Ker(f)$ is a subgroup of G.
11. Fix groups G, H, and a homomorphism $f : G \to H$. Let K be a subgroup of H. Prove that $S = \{g \in G : f(g) \in K\}$ is a subgroup of G.
12. Let G be an abelian group. Let $f : G \to G$ be defined via $f : x \mapsto x^{-1}$. Prove that f is a homomorphism.
13. Find a (non-abelian) group G, such that $f : G \to G$ defined via $f : x \mapsto x^{-1}$ is NOT a homomorphism.
14. Fix a group G, and an element $a \in G$. Define the centralizer of a as $C(a) = \{g \in G : ga = ag\}$. Prove that $C(a)$ is a subgroup of G.
15. Prove that $Z(G) = \bigcap_{a \in G} C(a)$, where $Z(G)$ denotes center, and $C(a)$ denotes centralizer.
16. Fix a group G, a subgroup H, and an element $a \in G$. Define $aHa^{-1} = \{aha^{-1} : h \in H\}$. Prove that aHa^{-1} is a subgroup of G.
17. Suppose G is a group that satisfies $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. Prove that G is abelian.
18. Suppose G is a group that satisfies $(ab)^3 = a^3b^3$ and $(ab)^5 = a^5b^5$ for all $a, b \in G$. Prove that G is abelian.
19. Fix a group G. For subgroups A, B, we define their product $AB = \{ab : a \in A, b \in B\}$.
    Suppose H, K are both subgroups of G that satisfy $HK = KH$. Prove that HK is a subgroup of G.
20. Consider a solid square prism, i.e. shoebox, with two identical square ends and four identical rectangular (but not square) sides, as pictured below. Color each face red, white, or blue. How many different ways are there to do this, up to physically possible isometries of the solid figure?