1. * For nonzero polynomial \( f(x) = a_n x^n + \cdots + a_1 x + a_0 \in \mathbb{Z}[x] \), define the content of \( f(x) \) as \( c(f) = \gcd(a_n, a_{n-1}, \ldots, a_1, a_0) \). We call \( f \) primitive if \( c(f) = 1 \). Let \( f(x), g(x) \in \mathbb{Z}[x] \). Suppose that \( f(x), g(x) \) are both primitive. Prove that their product \( f(x)g(x) \) is also primitive.

2. For nonzero \( f(x), g(x) \in \mathbb{Z}[x] \), prove that \( c(fg) = c(f)c(g) \).

3. * Let \( f(x) \in \mathbb{Z}[x] \). Suppose that there are non-units \( g(x), h(x) \in \mathbb{Q}[x] \) such that \( f(x) = g(x)h(x) \). Then there are \( g'(x), h'(x) \in \mathbb{Z}[x] \) such that \( f(x) = g'(x)h'(x) \) and \( \deg g(x) = \deg g'(x) \) (and also \( \deg h(x) = \deg h'(x) \)). Note: \( g'(x) \) is just another polynomial, not a derivative.

4. Fix \( a \in \mathbb{Z} \) and consider \( \phi_a : \mathbb{Z}[x] \rightarrow \mathbb{Z}[x] \) given by \( \phi_a : f(x) \mapsto f(x - a) \). Prove that if \( f(x) \) is reducible then \( \phi_a(f(x)) \) is reducible.

5. Use Eisenstein’s criterion (and Problem 4, if necessary) to prove that \( x^5 + 5x + 2 \) is irreducible in \( \mathbb{Q}[x] \).

6. Fix \( p \) prime, and consider the “natural map” \( \phi_p : \mathbb{Z}[x] \rightarrow \mathbb{Z}_p[x] \) given by \( \phi_p : a_n x^n + \cdots + a_1 x + a_0 \mapsto [a_n]_p x^n + \cdots + [a_1]_p x + [a_0]_p \). Prove that if \( p \nmid a_n \) and \( f(x) \) is primitive and reducible, then \( \phi_p(f(x)) \) is also reducible.

7. Use Problem 6 to prove that \( f(x) = x^3 + 5x + 4 \) is irreducible in \( \mathbb{Z}[x] \).

8. Set \( f(x) = 3x^3 + 4x^2 + 7x + 2 \). Show that this is reducible in \( \mathbb{Z}[x] \) but irreducible in \( \mathbb{Z}_3[x] \). Does this contradict problem 6?

9. Factor \( x^4 - 25 \) in \( \mathbb{Q}[x], \mathbb{R}[x] \), and \( \mathbb{C}[x] \).

10. Factor \( x^3 - ix^2 + 5x - 5i \) in \( \mathbb{C}[x] \).