1. Let \( R, S, T \) be rings, with \( S, T \) both subrings of \( R \). Suppose that \( S \) has the special property that for every \( s \in S \) and every \( r \in R \), we have both \( sr \in S \) and \( rs \in S \). Set \( S + T = \{ s + t : s \in S, t \in T \} \), a subset of \( R \). Prove that \( S + T \) is a subring of \( R \).

First, \( 0_R \in S \) and \( 0_T \in T \) (since \( S, T \) are subrings), so \( 0_R = 0_T + 0_R \in S + T \). Second, let \( x, x' \in S + T \). Then there are \( s, t, s', t' \in S, t, t' \in T \), and \( x = s + t, x' = s' + t' \). Now, we calculate \( x - x' = (s + t) - (s' + t') = (s - s') + (t - t') \). Since \( s - s' \in S \) and \( t - t' \in T \), we have \( x - x' \in S + T \). Lastly, we calculate \( xx' = (s + t)(s' + t') = (ss' + st' + ts' + tt') \). Since \( S \) is a ring, \( ss' \in S \). Similarly, since \( T \) is a ring, \( tt' \in T \). By our special property, both \( st' \) and \( ts' \) are in \( S \), so the sum \( ss' + st' + ts' \in S \). Hence \( xx' \in S + T \).

2. Consider the polynomial ring \( \mathbb{Z}_9[x] \), and the nine elements \( \{ 3x + 0, 3x + 1, \ldots, 3x + 8 \} \). Determine which are units and which are zero divisors.

Let’s first find units, by calculating \( (a + 3x)(b_0 + b_1x + \cdots + b_nx^n) = 1 \). Looking at the constant term, we have \( ab_0 \equiv 1 \pmod{9} \), so \( a \) is a unit modulo 9. This limits us to \( a \in \{ 1, 2, 4, 5, 7, 8 \} \).

A bit of trial and error shows that all six are units: \( (1 + 3x)(1 + 6x) \equiv (2 + 3x)(5 + 6x) \equiv (4 + 3x)(7 + 6x) \equiv (5 + 3x)(2 + 6x) \equiv (7 + 3x)(4 + 3x) \equiv (8 + 3x)(8 + 6x) \equiv 1 \pmod{9} \).

Now we look for zero divisors, by calculating \( (a + 3x)(b_0 + b_1x + \cdots + b_nx^n) = 0 \). Looking at the constant term, we have \( ab_0 \equiv 0 \pmod{9} \), so \( a \) is a zero divisor modulo 9. This limits us to \( a \in \{ 0, 3, 6 \} \). All three are zero divisors, as \( (a + 3x)(3) \equiv 3a + 0x \equiv 0 \pmod{9} \).

3. Consider the polynomial ring \( \mathbb{Z}_9[x] \), and the nine elements \( \{ 0x + 3, 1x + 3, \ldots, 8x + 3 \} \). Determine which are units and which are zero divisors.

For all nine elements, the constant term is 3; the argument in the preceding problem shows that none of these can be units, but some might be zero divisors. We have \( (0x + 3)(3) \equiv (3x + 3)(3) \equiv 0 \pmod{9} \), so these three are zero divisors. We now prove that the remaining six are neither units nor zero divisors.

Calculate \( (3 + ax)(b_0 + b_1x + \cdots + b_nx^n) = 0 \), and look at the \( x^{n+1} \) term. We must have \( ab_n \equiv 0 \pmod{9} \), so \( a \) must be a zero divisor modulo 9. However, none of \( \{ 1, 2, 4, 5, 7, 8 \} \) are zero divisors modulo 9, so none of \( \{ 1x + 3, 2x + 3, 4x + 3, 5x + 3, 7x + 3, 8x + 3 \} \) are zero divisors in \( \mathbb{Z}_9[x] \).

4. Let \( R \) be a ring, and \( k \in \mathbb{N} \). Define \( x^k R[x] = \{ x^k f(x) : f(x) \in R[x] \} \). Prove that \( x^k R[x] \) is a subring of \( R[x] \).

Certainly \( x^k R[x] \) is a subset of \( R[x] \), being polynomials (whose lowest-degree term is at least of degree \( k \) ). First, \( 0 = x^k 0 \), so the zero polynomial \( 0 \in x^k R[x] \). Now, let \( x^k f(x), x^k g(x) \in x^k R[x] \). We have \( x^k f(x) - x^k g(x) = x^k (f(x) - g(x)) \). Since \( f(x) - g(x) \in R[x] \), \( x^k f(x) - x^k g(x) \in x^k R[x] \). Lastly, we have \( x^k f(x)x^k g(x) = x^k (f(x)x^k g(x)) \). Since \( f(x)x^k g(x) \in R[x] \), we have \( x^k f(x)x^k g(x) \in x^k R[x] \).

5. Let \( F \) be a field. Determine explicitly which elements of \( F[x] \) are in the subring \( x^3 F[x] + x^5 F[x] \). (refer to exercises 1,4)
Exercises 1 and 4 prove that this object is a subring (provided we check the special property for either \(x^3F[x]\) or \(x^5F[x]\), which is not too hard to do). Suppose \(a(x) = a_0 + a_1x + \cdots + a_kx^k \in x^3F[x] + x^5F[x]\). Then there are polynomials \(b(x), c(x) \in F[x]\) with \(a(x) = x^3b(x) + x^5c(x) = (b_0x^3 + b_1x^4 + \cdots + b_mx^{3+m}) + (c_0x^5 + c_1x^6 + \cdots + c_nx^{5+n}) = b_0x^3 + b_1x^4 + \cdots\).

Note that this proves that \(a_0 = a_1 = a_2 = 0\), so in particular \(a(x) \in x^3F[x]\). Hence \(x^3F[x] + x^5F[x] \subseteq x^3F[x]\). But also \(x^3F[x] \subseteq x^3F[x] + x^5F[x]\), because for each \(x^3f(x) \in x^3F[x]\), we can write \(x^3f(x) = x^3f(x) + x^50 \in x^3F[x] + x^5F[x]\). Hence in fact \(x^3F[x] + x^5F[x] = x^3F[x]\).

6. Working in \(Q[x]\), find \(\gcd(a(x), b(x))\), for \(a(x) = x^3 + x^2 + x + 1\), \(b(x) = x^4 - 2x^2 - 3x - 2\).

\[
x^4 - 2x^2 - 3x - 2 = (x - 1)(x^3 + x^2 + x + 1) + (-2x^2 - 3x - 1)
\]
\[
x^3 + x^2 + x + 1 = \left(\frac{-1}{2}x + \frac{1}{4}\right)(-2x^2 - 3x - 1) + \left(\frac{5}{4}x + \frac{5}{4}\right)
\]
\[
-2x^2 - 3x - 1 = \left(\frac{-8}{5}x - \frac{4}{5}\right)\left(\frac{5}{4}x + \frac{5}{4}\right) + 0
\]

Hence \(\gcd(a, b)\) is the monic multiple of \(\frac{5}{4}x + \frac{5}{4}\), namely \(x + 1\).

7. Working in \(Z_2[x]\), find \(\gcd(a(x), b(x))\), for \(a(x) = x^3 + x^2 + x + 1\), \(b(x) = x^4 - 2x^2 - 3x - 2\).

We first note that \(b(x) = x^4 + x\), and calculate:

\[
x^4 + x = (x + 1)(x^3 + x^2 + x + 1) + (x + 1)
\]
\[
x^3 + x^2 + x + 1 = (x^2 + 1)(x + 1) + 0
\]

Hence \(\gcd(a, b) = x + 1\), which is already monic.

8. Working in \(Z_5[x]\), find \(\gcd(a(x), b(x))\), for \(a(x) = x^3 + x^2 + x + 1\), \(b(x) = x^4 - 2x^2 - 3x - 2\).

\[
x^4 - 2x^2 - 3x - 2 = (x - 1)(x^3 + x^2 + x + 1) + (3x^2 + 2x - 1)
\]
\[
x^3 + x^2 + x + 1 = (2x - 1)(3x^2 + 2x - 1) + 0
\]

Hence \(\gcd(a, b)\) is the monic multiple of \(3x^2 + 2x - 1\), namely \(2(3x^2 + 2x - 1) = x^2 + 4x + 3\).

9. Working in \(Q[x]\), let \(a(x) = x^2 - 5x + 6\), \(b(x) = x^3 - x^2 - 2x\). Find \(u(x), v(x)\) such that \(\gcd(a(x), b(x)) = a(x)u(x) + b(x)v(x)\).

\[
x^3 - x^2 - 2x = (x + 4)(x^2 - 5x + 6) + (12x - 24)
\]
\[
x^2 - 5x + 6 = \left(\frac{1}{12}x - \frac{1}{4}\right)(12x - 24)
\]

We solve for \(12x - 24\), getting \(12x - 24 = (x^3 - x^2 - 2x) + (-x - 4)(x^2 - 5x + 6)\). Now we normalize, to make the gcd monic, by multiplying by \(\frac{1}{12}\), getting \(x - 2 = (\frac{1}{12})(x^3 - x^2 - 2x) + (-\frac{1}{12}x - \frac{1}{3})(x^2 - 5x + 6)\). Hence the desired polynomials are \(u(x) = -\frac{1}{12}x - \frac{1}{3}\) and \(v(x) = \frac{1}{12}\).

10. Working in \(Z_3[x]\), let \(a(x) = x^2 - 5x + 6\), \(b(x) = x^3 - x^2 - 2x\). Find \(u(x), v(x)\) such that \(\gcd(a(x), b(x)) = a(x)u(x) + b(x)v(x)\).

Note that \(a(x) = x^2 + x\) and \(b(x) = x^3 + 2x^2 + x = (x^2 + x)(x + 1)\). Hence \(\gcd(a, b) = x^2 + x\), and \(x^2 + x = 1(x^2 + x) + 0(x^3 + 2x^2 + x)\), so we can take \(u(x) = 1, v(x) = 0\).