MATH 521A: Abstract Algebra
Homework 3: Due Sep. 27

1. Let $n, c \in \mathbb{N}$, and let $a, b \in \mathbb{Z}$. Suppose that $a \equiv b \pmod{n}$. Prove that $ac \equiv bc \pmod{n}$. Show that the converse does not hold [by giving an example of a, b, c, n where $ac \equiv bc \pmod{n}$ but $a \not\equiv b \pmod{n}$].
2. Find an integer x such that $x^2 \equiv 2 \pmod{31}$.
3. Which of $[0], [1], [2], [3], [4]$ is equal to $[2^{(3^{45})}]$, in $\mathbb{Z}_5$?
4. Let $a, b \in \mathbb{Z}$. Prove that $(a+b)^3 \equiv a^3 + b^3 \pmod{3}$. This is (a special case of) a theorem called the Freshman's Dream.
5. Let $n \in \mathbb{N}$, and $a, b \in \mathbb{Z}$. Suppose that $a \equiv b \pmod{n}$. Prove that $\gcd(a, n) = \gcd(b, n)$.
6. Write the $\oplus$-addition and $\odot$-multiplication tables of $\mathbb{Z}_9$.
7. For $\mathbb{Z}_9$, find the neutral additive element, the neutral multiplicative element, and all zero divisors. Be sure to justify your answer.
8. For $\mathbb{Z}_9$, find all the units and specify each inverse.

   We define $\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(a, b) : a \in \mathbb{Z}_3, b \in \mathbb{Z}_3\}$, the set of ordered pairs of elements, one from $\mathbb{Z}_3$ and one from another copy of $\mathbb{Z}_3$. We define operations in the natural way, i.e. componentwise:
   $(a, b) \oplus (a', b') = (a \oplus_3 a', b \oplus_3 b')$ and $(a, b) \odot (a', b') = (a \odot_3 a', b \odot_3 b')$.

9. Write the $\oplus$-addition and $\odot$-multiplication tables of $\mathbb{Z}_3 \times \mathbb{Z}_3$.
10. For $\mathbb{Z}_3 \times \mathbb{Z}_3$, find the neutral additive element, the neutral multiplicative element, and all zero divisors. Be sure to justify your answer.
11. For $\mathbb{Z}_3 \times \mathbb{Z}_3$, find all the units and specify each inverse.